

ON DISTRIBUTION FREE SKOROKHOD-MALLIAVIN CALCULUS

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ABSTRACT. The starting point of the current paper is a sequence of uncorrelated random variables. The distribution functions of these variables are assumed to be given but no assumptions on the types or the structure of these distributions are made. The above setting constitute the so called "distribution free" paradigm. Under these assumptions, a version of Skorokhod-Malliavin calculus is developed and applications to stochastic PDES are discussed.

1. INTRODUCTION

The theory and applications of Skorokhod-Malliavin calculus are well developed for Gaussian and Poisson processes, see, for example, A.V. Skorokhod [15], P. Malliavin [4], [5], G. Di Nunno et al [11], D. Nualart [9], M. Sanz-Sole [13].

In this paper we will introduce and investigate an extension of Skorokhod-Malliavin calculus to random fields generated by an *arbitrary* sequence $\Xi = (\xi_1, \xi_2, \dots)$ of square integrable and uncorrelated random variables on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let us assume that for every i , $\Pr(\xi_i < x) = F^i(x)$. These distribution functions *are given but their types are not specified*. Of course, some or all of these distribution functions could coincide.

The above setting constitute the so called "distribution free" paradigm. As the title suggests, our task is to develop a version of Malliavin-Skorokhod calculus in the distribution free setting. At the first glance, a rigorous implementation of the distribution free Skorokhod-Malliavin calculus might seem to be a long shot, however, it is not completely unexpected. Consider, for example, the following quotation from P.A. Meyer [12]: "The first and very important point is that, in the construction of multiple integrals, the Gaussian character of the process never appears".

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The main operators of Skorokhod-Malliavin calculus are Wick product (\diamond), Skorokhod integral (δ), and Malliavin derivative (\mathbb{D}). Skorokhod integral is an anti-derivative of Malliavin derivative and Wick product is an elementary but useful version of Skorokhod integral. In the current paper, these operators are defined and investigated in the distribution free setting under two natural assumptions (see **B1** and **B2** below) that hold, practically, for all standard types of random variables.

The *distribution free* version of Skorokhod-Malliavin calculus, developed in Sections 2-3, preserves the fundamental properties of the *classic* Malliavin-Skorokhod calculus. For example, the Itô-Skorokhod isometry

$$(1.1) \quad \mathbf{E} [|\delta(u)|^2] = \mathbf{E} \|u\|^2$$

$$(1.2) \quad \mathbf{E} [|\delta(u)|^2] = \mathbf{E} \|u\|^2 + \mathbf{E} [(\mathbb{D}u, \mathbb{D}u)]$$

where (1.1) holds for functions u adapted to the appropriate filtration and (1.2) holds for non-adapted random variables (see Proposition 4.12).

In the distribution free setting, it is natural to construct the *driving noise* \mathfrak{N} as follows:

$$\mathfrak{N} = \sum_k m_k \xi_k, \quad \xi_k \in \Xi,$$

where ξ_k are uncorrelated random variables, $\mathbf{E}\xi_k = 0$, $\mathbf{E}|\xi_k|^2 = 1$ and $\{m_k, k \geq 1\}$ is an orthogonal basis in some Hilbert space $\mathbf{H} = L_2(U, \mathcal{U}, \mu)$, where (U, \mathcal{U}, μ) is a σ -finite measure space.

Let J be the set of multiindices $\alpha = (\alpha_1, \alpha_2, \dots)$, such that for every k , $\alpha_k \in \mathbf{N}_0$, $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ and $|\alpha| = \sum_k \alpha_k$. We will construct a complete orthogonal system $\{\mathfrak{N}_\alpha, \alpha \in J\}$ in $L^2(\Omega, \sigma(\xi_k, k \geq 1), \mathbf{P})$. By construction, this basis is distribution free. The Cameron-Martin basis for Gaussian random fields is a particular case of the basis $\{\mathfrak{N}_\alpha, \alpha \in J\}$. The set of deterministic coefficients $\{X_\alpha = \mathbf{E}(X\mathfrak{N}_\alpha), \alpha \in J\}$ is often referred to as the *propagator* of the random variable/field X .

In Section 4, the distribution free Skorokhod-Malliavin "technology" is applied to the analysis of linear ordinary SDE as well as linear stochastic parabolic and elliptic SPDEs with additive or multiplicative distribution free noise. We will study these equations and their relations in adapted and non-adapted settings.

As an example, let us consider an adapted parabolic SPDE

$$(1.3) \quad u(t) = w + \int_0^t \mathcal{L}u(s)ds + \int_0^t \int_U [u(s)G(s, v) + f(s, v)] \diamond \mathfrak{N}(ds, dv),$$

where $\mathcal{L}u = a^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i}$. The propagator associated with (1.3) is given by following low-triangular system of deterministic PDEs for the

coefficients $u_0(t) = w_0$ and

$$(1.4) \quad \begin{cases} \partial_t u_\alpha(t) = \mathcal{L}u_\alpha + \sum_k \int_V m_k(u_{\alpha(k)}) G + f_{\alpha(k)} d\pi \\ u_\alpha(0) = w, \end{cases}$$

where $\alpha(k) = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots)$ and $|\alpha| > 0$. In contrast to stochastic PDE (1.3), the related propagator is a deterministic lower-triangular system. Therefore, under quite general assumptions, system (1.4) is solvable. Therefore, one can construct a distribution free "polynomial chaos" solution of equation (1.3) in the form $u = \sum_\alpha u_\alpha \mathfrak{N}_\alpha$.

Note that, due to its lower triangular structure, the propagator (1.4) can be solved sequentially. The latter bodes well to the efficiency of numerical implementation of the polynomial chaos solutions.

Similarly, under standard assumptions (e.g. positivity of the operator A), one can construct a polynomial chaos solution of the stationary equation

$$(1.5) \quad \mathbf{A}u + \sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n = f.$$

The propagator of this equation is given by the system

$$(1.6) \quad \begin{aligned} \mathbf{A}u_\alpha &= Ef \quad \text{if } |\alpha| = 0 \\ \mathbf{A}u_\alpha + \sum_{n \geq 1} \mathbf{M}_n u_{\alpha - \varepsilon_n} &= f_\gamma \quad \text{if } |\alpha| > 0, \end{aligned}$$

where ε_n is a multiindices with $|\varepsilon_n| = 1$ and the only non-zero component at the n^{th} coordinate.

Again, under standard assumptions (including positivity of the operator A), one can solve the propagator of the stationary equation and reconstruct the distribution free solution of (1.5) in the polynomial chaos form.

Note that, in both settings the triangular feature of the propagator is due to the linear structure of the underlying stochastic equations.

A much more difficult and nuanced problem is the existence of "distribution free" solutions for nonlinear stochastic equations driven by arbitrary noise. In general, one should not expect a "universal" answer, because the coefficients of expansions of a nonlinear function (e.g. a product) of random variables depends on the types of this random variables. For more detail and examples, see [16], [7].

Nevertheless, there exists at least one reasonably broad class of nonlinear SPDEs that fits into the distribution free paradigm. Specifically, SPDEs with the so-called Wick-polynomial nonlinearities belong to this class. Equation

$$(1.7) \quad \mathbf{A}u - u^{\diamond k} + \sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n = f$$

is an important example of a nonlinear equation from this class.

Note that the nonlinear part of equation (1.7) is a Wick power. By this reason, it is easy to see that the propagator of equation (1.7) is again a linear

lower triangular system given by

$$(1.8) \quad \mathbf{A}u_\alpha - \sum_{\kappa, \beta, \gamma: \kappa + \beta + \gamma = \alpha} u_\kappa u_\beta u_\gamma + \sum_{n \geq 1} \mathbf{M}_n u_{\alpha - \varepsilon_n} = f_\alpha$$

for all $\alpha \in J$. Due to its lower-triangular this system is uniquely solvable (under reasonable assumptions on the operators).

Wick products have also been used for designing unbiased approximations of stochastic Navier-Stokes equation (see [8] and the references therein). Specifically, in this types of models, the standard nonlinear term $\mathbf{v} \nabla \mathbf{v}$ was approximated by the Wick product $\mathbf{v} \diamond \nabla \mathbf{v}$. The stochastic Navier-Stokes equation with this correction is unbiased, that is the expectation of the solution coincides with the related deterministic Navier-Stokes equation.

Note that the first examples of systems with Wick nonlinearities were introduced and investigated in Euclidean Quantum Field theory (see e.g. [14]).

The most challenging problem in the analysis of distribution free noise is development of a distribution free calculus suitable for adapted and non-adapted "arbitrary" random fields. Not surprisingly, this subject constitutes the "foundation" of the paper. It is addressed in Section 3. This Section includes construction of *distribution free* multiple integrals, Skorokhod integral, Malliavin derivatives, and Wick exponent, as well as Ito-Skorokhod isometry. Linear SDEs and parabolic SPDEs are discussed in Section 4.02, 4.03. Linear and Wick-nonlinear elliptic (non-adapted) SPDEs are covered in Section 4.1. The Appendix deals with Wick products of multiple integrals.

2. DRIVING CYLINDRICAL RANDOM FIELDS

Let (U, \mathcal{U}, μ) be a σ -finite complete measure space. Let $\mathbf{H} = L_2(U, \mathcal{U}, \mu)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space.

Definition 1. A continuous linear functional \mathfrak{N} from \mathbf{H} to $L_2(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{E}[\mathfrak{N}(f)^2] = \|f\|_2^2$, $f \in \mathbf{H}$, is called a **driving** cylindrical random field on U .

It is assumed that $\mathbf{E}\mathfrak{N}(f) = 0$, $f \in \mathbf{H}$. Clearly, \mathfrak{N} is an isometric embedding of \mathbf{H} into $L^2(\Omega, \mathcal{F}, \mathbf{P})$.

Remark 1. If $f \in \mathbf{H}$ and $\{m_k\}$ is a complete orthonormal system in \mathbf{H} , then, obviously,

$$\mathfrak{N}(f) = \sum_k f_k \xi_k \text{ in } L_2(\Omega, \mathcal{F}, \mathbf{P}),$$

where $\xi_k = \mathfrak{N}(m_k)$ and $f_k = \int_U f m_k d\mu$ (note that $f = \sum_k f_k m_k$ in \mathbf{H}). Moreover,

$$\mathbf{E}[\mathfrak{N}(f) \mathfrak{N}(g)] = \int f g d\mu, \quad f, g \in \mathbf{H},$$

The noise (associated to \mathfrak{N}) is the formal series

$$\mathfrak{N}(v) = \sum_k m_k(v) \xi_k, v \in U.$$

We write

$$\mathfrak{N}(f) = \int_U \mathfrak{N}(v) f(v) \mu(dv) = \int_U f(v) \mathfrak{N}(dv) = \sum_k f_k \xi_k.$$

Remark 2. If (ξ_k) is an arbitrary sequence of centered uncorrelated r.v. in $L^2(\Omega, \mathbf{P})$, then for any $L_2(U, \mathcal{U}, \mu)$ the map

$$\mathfrak{N}(f) = \sum_k f_k \xi_k, f = \sum_k f_k m_k \in L^2(U, \mu),$$

is a driving cylindrical random field on U , that is any sequence of centered uncorrelated r.v. can define a generalized driving random field.

We shall introduce the following assumptions about \mathfrak{N} .

B1. Let $\xi_k = \mathfrak{N}(m_k), k \geq 1$. For each vector r.v. $(\xi_{i_1}, \dots, \xi_{i_n}), n \geq 1$, the moment generating function

$$M_{i_1 \dots i_n}(t) = M_{i_1 \dots i_n}(t_1, \dots, t_n) = \mathbf{E} \exp \{t_1 \xi_{i_1} + \dots t_n \xi_{i_n}\}$$

exists for all $t = (t_1, \dots, t_n)$ in some neighborhood of $0 \in \mathbf{R}^n$.

This assumption implies immediately that ξ_k have all the moments (see Theorem 5a, p. 57 in [17]). Let J be the set of all multiindices $\alpha = (\alpha_1, \dots)$ such that $\alpha_k \in \{0, 1, 2, \dots\}$ and $|\alpha| = \sum_k \alpha_k < \infty$. For $\alpha = (\alpha_k) \in J$ we denote

$$\xi^\alpha = \prod_k \xi_k^{\alpha_k}, \xi^0 = 1.$$

The following assumption is needed as well.

B2. Assume we are given an orthogonalization $\{\tilde{K}_\alpha, \alpha \in J\}$ of the system $O = \{\xi^\alpha, \alpha \in J\}$ such that for each n , $\{\tilde{K}_p, |p| \leq n\}$ spans the same linear subspace H_n as $\{\xi^p, |p| \leq n\}$ and for $|\alpha| = n + 1$,

$$\tilde{K}_\alpha = \xi^\alpha - \text{projection}_{H_n} \xi^\alpha$$

Set $\mathfrak{N}_\alpha = c_\alpha \tilde{K}_\alpha$ so that $\mathbf{E}[\mathfrak{N}_\alpha^2] = \alpha!$. The result is the complete orthogonal system $\{\mathfrak{N}_\alpha, \alpha \in J\}$. Obviously, $\mathfrak{N}_0 = 1$, and for $p \in J, p = \varepsilon_k$ ($\varepsilon_k \in J$ and has all components zeros except 1 as the k th component), $\mathfrak{N}_p = \mathfrak{N}_{\varepsilon_k} = \xi_k$.

Remark 3. a) If every $\xi_k = \mathfrak{N}(m_k)$ is bounded, then **B1** obviously holds.

b) The Hilbert space \mathbf{H} can be finite dimensional. In particular, if $U = \{1\}$ and μ is the Dirac measure δ_1 , then \mathbf{H} is one-dimensional, $m_1 = 1$. If $\xi = \mathfrak{N}(1)$, then the \mathfrak{N} -noise coincides with the r.v. $\mathfrak{N} = \xi$.

The following statement is almost obvious.

Proposition 1. *Assume **B1**, **B2** hold. Let $\mathcal{F}^0 = \sigma(\xi_k, k \geq 1) = \sigma(\mathfrak{N}(f), f \in \mathbf{H})$. Then $\{\mathfrak{N}_\alpha = \xi^{\odot \alpha}, \alpha \in J\}$ is a complete orthogonal system of $L_2(\Omega, \mathcal{F}^0, \mathbf{P})$: for each $\eta \in L_2(\Omega, \mathcal{F}^0, \mathbf{P})$,*

$$\eta = \sum_{\alpha} \eta_{\alpha} \mathfrak{N}_{\alpha},$$

where

$$\eta_{\alpha} = \frac{\mathbf{E}[\eta \mathfrak{N}_{\alpha}]}{\alpha!}.$$

Note that $\sum_{\alpha} \eta_{\alpha}^2 \alpha! = \mathbf{E}[\eta^2] < \infty$.

Proof. By Lemma 6, **B1** implies that any $\eta \in L_2(\mathcal{F}^0, \mathbf{P})$ can be approximated by a sequence of polynomials in $\xi^{\alpha}, \alpha \in J$. Therefore the orthogonalization described in **B2** defines a complete orthogonal system and $\{\mathfrak{N}_{\alpha}/\sqrt{\alpha!}, \alpha \in J\}$ is a CONS in $L_2(\mathcal{F}^0, \mathbf{P})$. \square

2.1. Examples.

2.1.1. *Driving cylindrical random fields and processes generated by independent r.v.'s.*

Example 1. *In the case of a single r.v. ξ as in Remark 3, $J = \{0, 1, 2, \dots\}$, the complete orthogonal system $\{\mathfrak{N}_n, n \geq 0\}$ of $L_2(\sigma(\xi), \mathbf{P})$ must coincide with the one obtained by Gram-Schmidt orthogonalization procedure. We set $\mathfrak{N}_0 = 1, \mathfrak{N}_1 = \xi$. If H_n be the subspace generated by $\{\mathfrak{N}_l, l \leq n\}$, then we take*

$$\tilde{K}_{n+1} = \xi^{n+1} - \text{projection}_{H_n}(\xi^{n+1}),$$

and set $\mathfrak{N}_{n+1} = c_{n+1} \tilde{K}_{n+1}$ so that $\mathbf{E}[\mathfrak{N}_{n+1}^2] = (n+1)!$.

Example 2. *Let ξ_k be a sequence of independent centered r.v. whose moment generating function exists in a neighborhood of zero. Assume $\mathbf{E}(\xi_k^2) = 1$.*

As already discussed in Remark 2, for any $L_2(U, \mathcal{U}, \mu)$ the map

$$\mathfrak{N}(f) = \sum_k f_k \xi_k, \quad f = \sum_k f_k m_k \in L^2(U, \mu),$$

*is a driving cylindrical random field on U . Obviously **B1** holds. For every k we apply orthogonalization procedure of Example 1 and construct \mathfrak{N}_l^k with $\mathbf{E}[(\mathfrak{N}_l^k)^2] = l!$. For a multiindex $\alpha \in J$, we set,*

$$(2.1) \quad \mathfrak{N}_{\alpha} = \prod_k \mathfrak{N}_{\alpha_k}^k.$$

*Note that $\mathbf{E}[\mathfrak{N}_{\alpha}^2] = \alpha!$ For any $\alpha \in J$ with $|\alpha| = n+1$, \mathfrak{N}_{α} is orthogonal to H_n and of the form $\xi^{\alpha} - l_{\alpha}$, where l_{α} is a linear combination of vectors in H_n , i.e. l_{α} is the orthogonal projection of ξ^{α} on H_n . Therefore **B1** is satisfied as well and $\{\mathfrak{N}_{\alpha}, \alpha \in J\}$ is a CONS in $L^2(\Omega, \sigma(\xi_k, k \geq 1), \mathbf{P})$.*

Example 3. Let ξ_k be a sequence independent centered r.v. that have all the moments and $\mathbf{E}(\xi_k^2) = 1$. Let (U, \mathcal{U}, μ) be a σ -finite measure space. Let $\{m_k, k \geq 1\}$ be a CONS in $\mathbf{H} = L_2(U, \mathcal{U}, \mu)$. We can define $\mathfrak{N} : L_2(U, \mathcal{U}, \mu) \rightarrow L_2(\Omega, \mathbf{P})$ as

$$(2.2) \quad \mathfrak{N}(f) = \sum_k f_k \xi_k, f = \sum_k f_k m_k \in \mathbf{H},$$

and the complete orthogonal system $\{\mathfrak{N}_\alpha, \alpha \in J\}$ as in Example 2. The \mathfrak{N} -noise is

$$\mathfrak{N}(x) = \sum_k m_k(x) \xi_k.$$

In particular, if $U = L^2([0, T])$ and $\{m_k, k \geq 1\}$ is a CONS on $L^2([0, T])$, then we can regard \mathfrak{N} in (2.2) as a stochastic process

$$(2.3) \quad \mathfrak{N}_t = \mathfrak{N}(\chi_{[0, t]}) = \sum_k \int_0^t m_k(s) ds \xi_k, 0 \leq t \leq T.$$

It has uncorrelated increments, $\mathbf{E}(\mathfrak{N}_t^2) = t$, $\mathbf{E}(\mathfrak{N}_t \mathfrak{N}_s) = t \wedge s$, $\mathbf{E}(\mathfrak{N}_t) = 0, t \geq 0$. For any continuous deterministic $f(t)$ on $[0, T]$:

$$\mathfrak{N}(f) = \lim_n \sum_{i=1}^n f(t_i) [\mathfrak{N}_{t_{i+1}} - \mathfrak{N}_{t_i}] \text{ in } L^2(\Omega, \mathbf{P}),$$

where $t_i = t_i^n, 0 \leq i \leq n$, is a partition of $[0, T]$ into n disjoint subintervals whose maximal size converges to zero as $n \rightarrow \infty$.

Some specific examples related to Example 3.

- For a standard normal $\xi \sim N(0, 1)$, the sequence \mathfrak{N}_n in Example 1 are Hermite polynomials: $\mathfrak{N}_n = \frac{d^n}{dz^n} p(z) \big|_{z=0}$ with

$$p(z) = \exp \left\{ z\xi - \frac{1}{2}z^2 \right\}, z \in \mathbf{R}.$$

With a sequence $\xi_k, k \geq 1$, of independent standard normals,

$$W_t = \sum_k \int_0^t m_k(s) ds \xi_k, 0 \leq t \leq T,$$

is a standard Wiener process (see [1]).

- Let ξ_k be i.i.d. such that $\mathbf{P}(\xi_k = 1) = \mathbf{P}(\xi_k = -1) = 1/2$. Then the driving process \mathfrak{N}_t in (2.3) is not Gaussian: for each $n > 1, u \in \mathbf{R}$,

$$\mathbf{E} \exp \left\{ \iota u \sum_{k=1}^n \int_0^t m_k(s) ds \xi_k \right\} = \prod_{k=1}^n \cos \left\{ u \int_0^t m_k(s) ds \right\}.$$

It is straightforward to find ($\xi_k^2 = 1$ a.s.) that for

$$\mathbf{E} \left[(W_t^n - W_s^n)^4 \right] \leq C |t - s|^2, s, t \in [0, T],$$

where $W_t^n = \sum_{k=1}^n \int_0^t m_k(s) ds \xi_k$, $0 \leq t \leq T$. Therefore, by the same arguments as in the Gaussian case (see [1]), W_t has a continuous in t modification.

- Let ξ_k be uniform i.i.d. on $[-\sqrt{3}, \sqrt{3}]$. Again, we have in (2.3) a non-Gaussian driving process with continuous paths:

$$\mathbf{E} \exp \left\{ \iota u \sum_{k=1}^n \int_0^t m_k(s) ds \xi_k \right\} = \prod_{k=1}^n \frac{\sin \left(u \int_0^t m_k(s) ds \right)}{\left(u \int_0^t m_k(s) ds \right)}.$$

The orthogonal basis consists of Legendre polynomials in this case. Let us consider the set of Legendre polynomials L_k on $[-1, 1]$ defined by the Rodrigues formula

$$(2.4) \quad L_n(\eta) = \frac{(-1)^n}{2^n n!} \frac{d^n}{d\eta^n} [(1 - \eta^2)^n].$$

For any n, m ,

$$(2.5) \quad \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} L_n\left(\frac{x}{\sqrt{3}}\right) L_m\left(\frac{x}{\sqrt{3}}\right) dx = \frac{1}{2} \int_{-1}^1 L_n(\eta) L_m(\eta) d\eta = \frac{1}{(2n+1)} \delta_{nm}.$$

Let $\tilde{B}_n = L_n(x/\sqrt{3})$. To make it "standard" ($\mathbf{E}(B_n^2) = n!$), we set

$$B_0 = 1, B_n = \sqrt{n!(2n+1)} \tilde{B}_n.$$

In this case $\mathfrak{N}_k^n = \xi_k^{\otimes n} = B_n(\xi_k)$, $k \geq 1, n \geq 0$ and we get the whole system by (2.1).

A sequence of independent driving random fields can be used to construct a new one.

2.1.2. Poisson random fields. Let N be Poisson random measure on U with μ as its Levy measure and $\tilde{N} = N - \mu$. We have isometry:

$$\mathbf{E} [\tilde{N}(\varphi)^2] = \int_U \varphi^2 d\mu, \varphi \in L_2(U, \mu).$$

Let $m_k(x)$ be a CONS in $L_2(U, \nu)$ such that all m_k and $\int_U |m_k(v)| d\mu$ are bounded. In general, $\xi_k = \tilde{N}(m_k)$ are not independent.

Let \mathcal{Z} be the set of all real-valued sequences $z = (z_k)$ such that only the finite number of z_k is not zero. For $z \in \mathcal{Z}$ set, $m = m_z = \sum_k z_k m_k$. Under the assumptions above, the moment generating function

$$\begin{aligned} M(z) &= \mathbf{E} \exp \left\{ \tilde{N}(m_z) \right\} \\ &= \exp \left\{ \int_U \left[e^{m_z(v)} - 1 - m_z(v) \right] \mu(dv) \right\} \end{aligned}$$

exists and Assumption **B1** is satisfied.

Charlier polynomials

For small z , let

$$p(z) = \exp \left\{ \int_U \ln[1 + m_z(v)] N(dv) - \int_U m_z(v) \mu(dv) \right\}.$$

For $\alpha \in J$, we define Charlier polynomials as

$$C_\alpha = \frac{\partial^\alpha}{\partial z^\alpha} p(z)|_{z=0}, \alpha \in J.$$

For example, if $|\alpha| = 1, \alpha = \varepsilon_k$, then $C_\alpha = \tilde{N}(m_k)$; If $|\alpha| = 2, \alpha = \varepsilon_{k_1} + \varepsilon_{k_2}$, then

$$\begin{aligned} C_\alpha &= N(m_{k_1}) N(m_{k_2}) - N(m_{k_1} m_{k_2}) - N(m_{k_1}) \mu(m_{k_2}) \\ &\quad - N(m_{k_2}) \mu(m_{k_1}) + \mu(m_{k_1}) \mu(m_{k_2}), \end{aligned}$$

where $N(\phi) = \int_U \phi(x) N(dx), \mu(\phi) = \int_U \phi(x) \mu(dx)$. Recall $\varepsilon_k = (\alpha_i) \in J$ and has all components zero except $\alpha_k = 1$.

It is a standard fact that Charlier polynomials are orthogonal: $\mathbf{E} C_\alpha C_{\alpha'} = \alpha!$ if $\alpha = \alpha'$ and zero otherwise. In general and contrary to the Hermite polynomials for the Gaussian random variables, they are not polynomials of Poisson random variables.

We define driving Poisson noise on U as $\mathfrak{N}(f) = \tilde{N}(f), f \in L^2(U, d\mu)$. From our general considerations above, it satisfies **B1**, **B2** with $\mathfrak{N}_\alpha = C_\alpha, \alpha \in J$.

2.2. Generalized random variables and processes. Let E be a topological vector space and

$$\mathcal{D}(E) = \left\{ v = \sum_{\alpha} v_{\alpha} \mathfrak{N}_{\alpha} : v_{\alpha} \in E \text{ and only finite number of } v_{\alpha} \text{ are not zero} \right\}.$$

Definition 2. A generalized \mathcal{D} -random variable with values in E with Borel σ -algebra is a formal series $u = \sum_{\alpha} u_{\alpha} \xi_{\alpha}$, where $u_{\alpha} \in E$.

Denote the vector space of all generalized \mathcal{D} -random variables by $\mathcal{D}' = \mathcal{D}'(E)$. If $E = \mathbf{R}$ we write simply $\mathcal{D}, \mathcal{D}'$. The elements of \mathcal{D} are the test random variables for \mathcal{D}' . We define the action of a generalized random variable $u \in \mathcal{D}'(E)$ on the test random variable $v \in \mathcal{D}$ by $\langle u, v \rangle = \sum_{\alpha} v_{\alpha} u_{\alpha}$. If Y is Hilbert and $u \in \mathcal{D}'(Y), v \in \mathcal{D}(E)$, then $\langle u, v \rangle = \sum_{\alpha} (v_{\alpha}, u_{\alpha})_Y$.

For a sequence $u^n \in \mathcal{D}'$ and $u \in \mathcal{D}'$, we say that $u^n \rightarrow u$, if for every $v \in \mathcal{D}$, $\langle u, v^n \rangle \rightarrow \langle u, v \rangle$. This implies that $u^n = \sum_{\alpha} u_{\alpha}^n \mathfrak{N}_{\alpha} \rightarrow u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha}$ if and only if $u_{\alpha}^n \rightarrow u_{\alpha}$ as $n \rightarrow \infty$ for all α .

Remark 4. If $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'(E)$, F is a vector space and $f : E \rightarrow F$ is a linear map, then we define

$$f(u) = \sum_{\alpha} f(u_{\alpha}) \mathfrak{N}_{\alpha} \in \mathcal{D}'(F).$$

Definition 3. An E -valued generalized \mathcal{D} -field on a measurable space (B, \mathcal{B}) is a $\mathcal{D}'(E)$ -valued function on B such that for each $x \in B$,

$$u(x) = \sum_{\alpha} u_{\alpha}(x) \mathfrak{N}_{\alpha} \in \mathcal{D}'(E),$$

where $u_{\alpha}(x)$ are deterministic measurable E -valued functions on B .

We denote the linear space of all such fields by $\mathcal{D}'(B; E)$. If E is a topological vector space and a generalized \mathcal{D} -field $u(x)$ is continuous on B we write $u \in C\mathcal{D}'(B; E)$ (note that $u(x)$ is continuous if and only if all coefficient functions u_{α} on B are continuous). In particular, if $B = [0, T]$, we say $u(t)$ is a generalized \mathcal{D} -process. If there is no room for confusion, we will often say \mathcal{D} -process (\mathcal{D} -random variable) instead of generalized \mathcal{D} -process (generalized \mathcal{D} -random variable).

If (B, \mathcal{B}, κ) is a measure space and E is a normed vector space, we denote

$$\begin{aligned} & L_2(\mathcal{D}'(B; E), \kappa) \\ &= \{u(x) = \sum_{\alpha} u_{\alpha}(x) \mathfrak{N}_{\alpha} \in \mathcal{D}'(B; E) : \int_B |u_{\alpha}(x)|_E^2 d\kappa < \infty, \alpha \in J\}. \end{aligned}$$

For $u(t) = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha} \in L_1(\mathcal{D}'([0, T], E))$ we define $\int_0^t u(s) ds, 0 \leq t \leq T$, in $\mathcal{D}'([0, T]; E)$ by

$$\int_0^t u(s) ds = \sum_{\alpha} \left(\int_0^t u_{\alpha}(s) ds \right) \mathfrak{N}_{\alpha}, 0 \leq t \leq T.$$

If $u(t) = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha} \in \mathcal{D}'([0, T]; E)$, then $u(t)$ is differentiable in t if and only if $u_{\alpha}(t)$ are differentiable in t . In that case,

$$\frac{d}{dt} u(t) = \dot{u}(t) = \sum_{\alpha} \dot{u}_{\alpha}(t) \mathfrak{N}_{\alpha} \in \mathcal{D}'([0, T], E).$$

3. Distribution Free SKOROKHOD-MALIAVIN Calculus

Let $\mathbf{H}^{\otimes n} = L_2(U^n, \mathcal{U}^{\otimes n}, \mu_n)$, where $U^n = U \times \dots \times U$ n -times, $\mu_n = \mu^{\otimes n}$. Let $\hat{\mathbf{H}}^{\otimes n}$ be the symmetric part of $\mathbf{H}^{\otimes n}$: it is the set of all symmetric μ_n -square integrable functions on U^n . For $\alpha \in J$ with $|\alpha| = n$ we define its characteristic set $K_{\alpha} = \{k_1, \dots, k_n\}$ with each k represented in it by α_k copies. Let \mathcal{G}_n be the group of permutations of $\{1, \dots, n\}$ and

$$E_{\alpha} = \sum_{\sigma \in \mathcal{G}_n} m_{k_{\sigma(1)}} \otimes \dots \otimes m_{k_{\sigma(n)}}.$$

Then

$$e_{\alpha} = \frac{E_{\alpha}}{\sqrt{\alpha! |\alpha|!}}, \alpha \in J, |\alpha| = n,$$

is a CONS of the symmetric part of $\mathbf{H}^{\otimes n}$. If $|p| = 1$ with k th component non zero, then $E_p = e_p = m_k$.

3.1. Multiple integrals, Wick product and Skorokhod integral. Multiple integrals

Now we construct multiple integrals with respect to \mathfrak{N} on the symmetric part $\mathbf{H}^{\hat{\otimes} n}$ ($\mathbf{H}^{\hat{\otimes} 0} = \mathbf{R}$). Set for $|\alpha| = n \geq 1$,

$$I_n(E_\alpha) = n! \mathfrak{N}_\alpha.$$

Let Y be a Hilbert space and denote $\mathbf{H}^{\hat{\otimes} n}(Y)$ the space of all Y -valued symmetric functions $v = \sum_{|\alpha|=n} v_\alpha E_\alpha$ on U^n such that

$$|v|_{\mathbf{H}^{\hat{\otimes} n}(Y)}^2 = \int_{U^n} |v(r)|_Y^2 d\mu_n = \sum_{\alpha} |v_\alpha|_Y^2 |\alpha|! \alpha! < \infty.$$

For $v = \sum_{|\alpha|=n} v_\alpha E_\alpha \in \mathbf{H}^{\hat{\otimes} n}(Y)$, we define its n -tuple integral as

$$I_n(v) = \sum_{|\alpha|=n} v_\alpha I_n(E_\alpha) = \sum_{|\alpha|=n} v_\alpha n! \mathfrak{N}_\alpha,$$

and $I_0(c) = c, c \in \mathbf{R}$. Note that

$$\int_{U^n} \left(\sum_{|\alpha|=n} v_\alpha E_\alpha \right)^2 d\mu_n = \sum_{|\alpha|=n} v_\alpha^2 n! \alpha!$$

and

$$\mathbf{E} [|I_n(v)|_Y^2] = n!^2 \sum_{|\alpha|=n} |v_\alpha|_Y^2 \alpha! = n! |v|_{\mathbf{H}^{\hat{\otimes} n}(Y)}^2.$$

Let $\mathcal{S}(Y)$ be the space of all finite linear combinations $\sum_k \frac{I_k(F_k)}{k!}$ with $F_k \in \mathbf{H}^{\hat{\otimes} k}(Y)$.

Definition 4. A generalized \mathcal{S} -random variable is a formal sum

$$u = \sum_k \frac{I_k(F_k)}{k!} \text{ with } F_k \in \mathbf{H}^{\hat{\otimes} k}(Y).$$

We denote the set of all generalized \mathcal{S} -random variables by $\mathcal{S}'(Y)$.

The action of $u \in \mathcal{S}'(Y)$ on $v \in \mathcal{S}(Y)$ is defined as

$$\langle u, v \rangle = \sum_k \int_{U^k} (u_k, v_k)_Y d\mu,$$

where $u = \sum_k I_k(u_k)/k!, v = \sum_k I_k(v_k)/k!$ with $u_k, v_k \in \mathbf{H}^{\hat{\otimes} k}(Y)$.

Definition 5. A generalized \mathcal{S} -field on a measurable space (B, \mathcal{B}) is a $\mathcal{S}'(Y)$ -valued function on B such that for each $x \in B$,

$$u(x) = \sum_n \frac{I_n(F_n(x))}{n!} \in \mathcal{S}'(Y),$$

where $F_n(x) = F_n(x; v_1, \dots, v_n)$ are deterministic measurable $\mathbf{H}^{\hat{\otimes} n}(Y)$ -valued functions on B .

We denote the linear space of all such fields by $\mathcal{S}'(B; E)$. If a generalized \mathcal{S} -field $u(x)$ is continuous on B we write $u \in C\mathcal{S}'(B; E)$ (note that $u(x)$ is continuous if and only if all coefficient functions F_n on B (as $\mathbf{H}^{\hat{\otimes} k}(Y)$ -valued functions) are continuous). In particular, if $B = [0, T]$, we say $u(t)$ is a generalized \mathcal{S} -process.

Remark 5. Obviously, $\mathcal{D} \subseteq \mathcal{S}(\mathbf{R})$ and $\mathcal{S}'(Y) \subseteq \mathcal{D}'(Y)$. In fact,

$$\mathcal{S}'(Y) = \left\{ u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'(Y) : \sum_{|\alpha|=n} |u_{\alpha}|_Y^2 \alpha! < \infty \ \forall n \geq 1 \right\}.$$

Indeed, if $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha}$ with $\sum_{|\alpha|=n} |u_{\alpha}|_Y^2 \alpha! < \infty, n \geq 1$, then

$$u_n = \sum_{|\alpha|=n} u_{\alpha} E_{\alpha} \in \mathbf{H}^{\hat{\otimes} n}(Y)$$

and

$$u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} u_{\alpha} \mathfrak{N}_{\alpha} = \sum_{n=0}^{\infty} \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y).$$

Note that for $|\alpha| = n$,

$$u_{\alpha} = \frac{1}{\alpha! n!} \int_{U^n} u_n(v) E_{\alpha}(v) d\mu_n, n \geq 1.$$

For $n \geq 0$, let $\mathcal{E}\mathbf{H}^{\hat{\otimes} n}$ be the space of all finite linear combinations of $E_{\alpha}, |\alpha| = n$. The following statement provides some insight about the transition from n -tuple integral to an integral on U^{n+1} .

Proposition 2. Let $f \in \mathcal{E}\mathbf{H}^{\hat{\otimes} n}, g \in \mathcal{E}\mathbf{H}$, $f \otimes g = f(z)g(v), z \in U^n, v \in U$, and let $\widetilde{f \otimes g}$ be the standard symmetrization of $f \otimes g$. Then

$$I_{n+1}(\widetilde{f \otimes g}) = I_n(f) I_1(g) - \text{projection}_{H_n} [I_n(f) I_1(g)].$$

Proof. Let

$$f = \sum_{|p|=n} f_p E_p, g = \sum_{|p|=1} g_p E_p.$$

Then

$$\begin{aligned} fg &= \sum_{p, p'} f_p g_{p'} E_p E_{p'}, \\ \widetilde{f \otimes g} &= \sum_{p, p'} f_p g_{p'} \widetilde{E_p E_{p'}} = \frac{1}{n+1} \sum_{p, p'} f_p g_{p'} E_{p+p'} \end{aligned}$$

and

$$\begin{aligned} I_{n+1}(\widetilde{f \otimes g}) &= \frac{1}{n+1} \sum_{p, p'} f_p g_{p'} I_{n+1}(E_{p+p'}) = n! \sum_{p, p'} f_p g_{p'} \mathfrak{N}_{p+p'}, \\ I_n(f) &= n! \sum_p f_p \mathfrak{N}_p, I_1(g) = \sum_{p'} g_{p'} \mathfrak{N}_{p'} \end{aligned}$$

Since

$$\mathfrak{N}_{p+p'} = \mathfrak{N}_p \mathfrak{N}_{p'} - \text{projection}_{H_n} [\mathfrak{N}_p \mathfrak{N}_{p'}],$$

it follows that

$$\begin{aligned} I_{n+1} \left(\widetilde{f \otimes g} \right) &= n! \sum_{p,p'} f_p g_{p'} [\mathfrak{N}_p \mathfrak{N}_{p'} - \text{projection}_{H_n} (\mathfrak{N}_p \mathfrak{N}_{p'})] \\ &= I_n(f) I_1(g) - \text{projection}_{H_n} [I_n(f) I_1(g)]. \end{aligned}$$

□

Remark 6. If $U = [0, T]$, $d\mu = dt$ and $m_1 = \chi_{(a,b)}$ (an interval of unit length), then according to Proposition 2, the "measure" of the square

$$I_2 \left(\chi_{(a,b)}^{\otimes 2} \right) = I_1 \left(\chi_{(a,b)} \right)^2 - \text{projection}_{H_1} \left[I_1 \left(\chi_{(a,b)} \right)^2 \right].$$

Wick product and Skorokhod integral. We define Wick product

$$\mathfrak{N}_\alpha \diamond \mathfrak{N}_\beta = \mathfrak{N}_{\alpha+\beta}, 1 \diamond \mathfrak{N}_\alpha = \mathfrak{N}_\alpha, \alpha, \beta \in J.$$

For $u = \sum_\alpha u_\alpha \mathfrak{N}_\alpha, v = \sum_\alpha v_\alpha \mathfrak{N}_\alpha \in \mathcal{D}'(E)$ (E is Hilbert),

$$u \diamond v = \sum_\alpha \sum_{\beta \leq \alpha} (u_\beta, v_{\alpha-\beta})_E \mathfrak{N}_\alpha.$$

For a generalized random field on $u = \sum_\alpha u_\alpha \mathfrak{N}_\alpha \in \mathcal{D}'(\mathbf{H}(Y))$, we define its Skorokhod integrals:

$$\delta_p(u) = \int_U u(v) \diamond \mathfrak{N}_p(dv) = \sum_\alpha \int_U a_\alpha(v) E_p(v) d\mu \mathfrak{N}_{\alpha+p}, |p| = 1,$$

and

$$\begin{aligned} \delta(u) &= \int_U u(v) \diamond \mathfrak{N}(dv) = \int_U u(v) \diamond \dot{\mathfrak{N}}(v) \mu(dv) = \sum_{|p|=1} \delta_p(u) \\ &= \sum_\alpha \sum_{|p|=1} \int_U a_\alpha(x) E_p(x) d\mu \mathfrak{N}_{\alpha+p} = \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_U a_{\alpha-p}(x) E_p(x) d\mu \mathfrak{N}_\alpha. \end{aligned}$$

Note that for a deterministic $u = u \mathfrak{N}_0 \in \mathbf{H}(Y)$,

$$\delta_{\varepsilon_k}(u) = \int_U u(v) m_k(v) d\mu \mathfrak{N}_{\varepsilon_k}, \delta(u) = \sum_k \int_U u(v) m_k(v) d\mu \mathfrak{N}_{\varepsilon_k} = \mathfrak{N}(u).$$

Now, we describe Skorokhod integral in terms of the multiple integrals I_n . We show that $\delta : \mathcal{S} \rightarrow \mathcal{S}$ and $\delta : \mathcal{S}' \rightarrow \mathcal{S}'$.

Proposition 3. Let

$$u = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(u_n) \in \mathcal{S}'(L^2(U, d\mu)),$$

i.e.,

$$u_n = u_n(v; v_1, \dots, v_n) = \sum_{|\alpha|=n} u_\alpha(v) E_\alpha(v_1, \dots, v_n),$$

with

$$\sum_{|\alpha|=n} \int |u_\alpha(v)|^2 d\mu \alpha! < \infty \quad \forall n.$$

Then

$$\delta(u) = \int u(v) \diamond \mathfrak{N}(dv) = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\tilde{F}_n),$$

where \tilde{F}_n is the standard symmetrization of F_n on U^{n+1} .

Proof. According to Remark 5, for $|\alpha| = n$,

$$u_\alpha(v) = \frac{1}{\alpha!n!} \int_{U^n} u_n(v; v') E_\alpha(v') \mu_n(dv')$$

and

$$u = \sum_{\alpha} u_\alpha(v) \mathfrak{N}_\alpha.$$

Note that

$$\int_{U \times U^n} u_n(v; v')^2 d\mu_{n+1} < \infty$$

and

$$\begin{aligned} \int_U u_\alpha(v) E_p(v) d\mu &= \frac{1}{\alpha!n!} \int_U \int_{U^n} u_n(r; v') E_p(r) \mu(dr) E_\alpha(v') \mu_n(dv') \\ &= \frac{1}{\sqrt{\alpha!n!}} \int_{U^n} \int_U u_n(r; v') E_p(r) \mu(dr) e_\alpha(v') \mu_n(dv') \end{aligned}$$

and

$$u_n(v, v') = \sum_{|\alpha|=n, |p|=1} \int_U u_\alpha(r) E_p(r) d\mu E_p(v) E_\alpha(v').$$

Therefore, the standard symmetrization of $u_n(v; \cdot, v')$, $v \in U, v' = (v_1, \dots, v_n) \in U^n$, is

$$\tilde{u}_n(v, v') = \sum_{|\alpha|=n+1, |p|=1} \int_U u_{\alpha-p}(r) E_p(r) d\mu \frac{n!}{(n+1)!} E_\alpha(v, v').$$

By definition of the Skorokhod integral,

$$\begin{aligned} \delta(u) &= \sum_{\alpha} \sum_{|p|=1} \int_U a_\alpha(r) E_p(r) d\mu \mathfrak{N}_{\alpha+p} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \sum_{|p|=1} \int_U a_\alpha(r) E_p(r) d\mu \frac{I_{n+1}(E_{\alpha+p})}{(n+1)!} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n+1} \sum_{|p|=1} \int_U a_{\alpha-p}(r) E_p(r) d\mu \frac{I_{n+1}(E_\alpha)}{(n+1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\tilde{F}_n), \end{aligned}$$

and the statement follows. \square

Multiple Skorokhod integrals

For symmetric $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'(\mathbf{H}^{\hat{\otimes} n})$, with $u_{\alpha} \in \mathbf{H}^{\hat{\otimes} n}$, and $|p| = n$, we define

$$\begin{aligned}\delta_p(u) &= \sum_{\alpha} \int_{U^n} u_{\alpha} \frac{E_p}{p!} d\mu_n \mathfrak{N}_{\alpha+p}, \\ \delta^n(u) &= \sum_{|p|=n} \delta_p(u) = \sum_{\alpha} \sum_{|p|=n} \int_{U^n} u_{\alpha} \frac{E_p}{p!} d\mu_n \mathfrak{N}_{\alpha+p}.\end{aligned}$$

Let $\delta^0(u) = u_0$. Note that for a deterministic $u(x_1, \dots, x_n) = u(x_1, \dots, x_n) \mathfrak{N}_0$ in $\mathbf{H}^{\hat{\otimes} n}$,

$$\begin{aligned}\delta^n(u) &= \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n \mathfrak{N}_p = \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n I_n(E_p) / n!, \\ \delta^n(E_p) &= \int_{U^n} E_p \frac{E_p}{p!} d\mu_n \mathfrak{N}_p = n! \mathfrak{N}_p = I_n(E_p), \delta^n(u) = I_n(u).\end{aligned}$$

It is easy to show that for $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'(\mathbf{H}^{\hat{\otimes} n})$, $n \geq 1$, (with $u_{\alpha} \in \mathbf{H}^{\hat{\otimes} n}$), $\delta^n(u) = \delta(\delta^{n-1}(\tilde{u}(v)))$, where

$$\tilde{u}(v) = \tilde{u}(v; v_1, \dots, v_{n-1}) = u(v, v_1, \dots, v_{n-1}), v, v_i \in U.$$

Remark 7. In the framework of a single r.v. ξ (see Remark 3),

$$\delta^k(\mathfrak{N}_n) = \mathfrak{N}_{n+k}, k \geq 1.$$

Lemma 1. For a deterministic $u = u \mathfrak{N}_0$ in $\mathbf{H}^{\hat{\otimes} n}$,

$$\mathbf{E}[\delta^n(u)^2] = n! \int_{U^n} |u|^2 d\mu_n.$$

Proof. Indeed,

$$\begin{aligned}\delta^n(u) &= \sum_{|p|=n} \int_{U^n} u \frac{E_p}{p!} d\mu_n \mathfrak{N}_p, \\ \mathbf{E}[\delta^n(u)^2] &= \sum_{|p|=n} \left(\int_{U^n} u \frac{E_p}{p!} d\mu_n \right)^2 p! = n! \int_{U^n} |u|^2 d\mu_n.\end{aligned}$$

□

Remark 8. We can rewrite Proposition 1 using multiple integrals. For each $\eta \in L_2(\Omega, \mathcal{F}^0, \mathbf{P})$,

$$\eta = \sum_{\alpha} \eta_{\alpha} \mathfrak{N}_{\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha|=n} \eta_{\alpha} I_n(E_{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\eta_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(\eta_n)$$

with

$$\eta_{\alpha} = \frac{\mathbf{E}[\eta \mathfrak{N}_{\alpha}]}{\alpha!},$$

and

$$\eta_n = \eta_n(v) = \sum_{|\alpha|=n} \eta_\alpha E_\alpha(v).$$

Note that

$$\eta_\alpha = \frac{1}{\alpha!n!} \int_{U^n} F_n(v) E_\alpha(v) d\mu_n, \alpha \in J.$$

3.2. Malliavin derivative. We define

$$\mathbb{D}\mathfrak{N}_\alpha = \sum_{|p|=1, p \leq \alpha} \frac{\alpha!}{(\alpha-p)!} \mathfrak{N}_{\alpha-p} E_p(v) = \sum_{\gamma} \sum_{|p|=1, \gamma+p=\alpha} \frac{(\gamma+p)!}{\gamma!} E_p(v) \mathfrak{N}_\gamma.$$

For $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}$,

$$\begin{aligned} \mathbb{D}_k u &= \sum_{|\alpha| \geq 1} \alpha_k u_{\alpha} \mathfrak{N}_{\alpha(k)} m_k(v), \mathbb{D}_p u = \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha-p)!} u_{\alpha} \mathfrak{N}_{\alpha-p} E_p(v), |p|=1, \\ \mathbb{D}u &= \sum_{|p|=1} \mathbb{D}_p u = \sum_{\alpha \geq p} \sum_{|p|=1} \frac{\alpha!}{(\alpha-p)!} u_{\alpha} \mathfrak{N}_{\alpha-p} E_p(v) \\ &= \sum_{\alpha} \sum_{|p|=1} \frac{(\alpha+p)!}{\alpha!} u_{\alpha+p} E_p(v) \mathfrak{N}_{\alpha}. \end{aligned}$$

In a standard way we define the higher order Malliavin derivatives: for $u = \sum_{\alpha} a_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}$,

$$\begin{aligned} \mathbb{D}_p^n u &= \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha-p)!} u_{\alpha} \mathfrak{N}_{\alpha-p} \frac{E_p(v_1, \dots, v_n)}{p!}, |p|=n, \\ \mathbb{D}^n u &= \mathbb{D}_p^n u = \sum_{|p|=n} \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha-p)! p!} u_{\alpha} \mathfrak{N}_{\alpha-p} E_p(v_1, \dots, v_n) \\ &= \sum_{\alpha} \sum_{|p|=n} \frac{(\alpha+p)!}{\alpha! p!} u_{\alpha+p} E_p(v_1, \dots, v_n) \mathfrak{N}_{\alpha}. \end{aligned}$$

We define Malliavin derivative for multiple integrals as well.

Proposition 4. Let $v = \sum_{|\alpha|=n} v_{\alpha} E_{\alpha} \in \mathbf{H}^{\otimes n}(Y)$ has only finite number of $v_{\alpha} \neq 0$. Then

$$\mathbb{D}I_n(u) = nI_{n-1}(u(\cdot, t)), t \in U.$$

Proof. By definition,

$$I_n(v) = \sum_{|\alpha|=n} v_{\alpha} I_n(E_{\alpha}) = \sum_{|\alpha|=n} v_{\alpha} n! \mathfrak{N}_{\alpha} \in \mathcal{D}.$$

Since

$$v_{\alpha} = \frac{1}{\alpha!n!} \int_{U^n} v E_{\alpha} d\mu_n,$$

and for $t, v_1, \dots, v_n \in U$,

$$v(t, v_1, \dots, v_{n-1}) = \sum_{|p|=1} \int_U v(t', v_1, \dots, v_{n-1}) E_p(t') d\mu_{E_p}(t)$$

(it is a finite sum), its Malliavin derivative $(\{(\alpha, p) : |\alpha + p| = n, |p| = 1\} = \{(\alpha, p) : |\alpha| = n-1, |p| = 1\})$ is

$$\begin{aligned} & \mathbb{D}I_n(v) \\ &= \sum_{\alpha} \sum_{|p|=1, |\alpha+p|=n} \frac{(\alpha+p)!}{\alpha!} v_{\alpha+p} n! E_p(v) \mathfrak{N}_{\alpha} = \sum_{|\alpha|=n-1} \sum_{|p|=1} \frac{(\alpha+p)!}{\alpha!} v_{\alpha+p} n! E_p(v) \mathfrak{N}_{\alpha} \\ &= \sum_{|\alpha|=n-1} \sum_{|p|=1} \frac{1}{\alpha!} \int v E_{\alpha+p} d\mu_n E_p(v) \mathfrak{N}_{\alpha} = \sum_{|\alpha|=n-1} \sum_{|p|=1} n \int v \frac{E_{\alpha}}{\alpha!} E_p d\mu_n E_p(t) \mathfrak{N}_{\alpha} \\ &= \sum_{|\alpha|=n-1} n(n-1)! \int v(t, \cdot) \frac{E_{\alpha}}{\alpha!(n-1)!} d\mu_{n-1} \mathfrak{N}_{\alpha} = \sum_{|\alpha|=n-1} n(n-1)! v_{\alpha}(t, \cdot) \mathfrak{N}_{\alpha} \\ &= n I_{n-1}(v(\cdot, t)). \end{aligned}$$

We used here that

$$\widetilde{E_{\alpha} E_p} = \frac{|\alpha|!}{|\alpha+p'|!} E_{\alpha+p} = \frac{1}{n} E_{\alpha+p},$$

where \widetilde{f} is the symmetrization of f . □

As suggested by Proposition 4, for an arbitrary $v = \sum_{|\alpha|=n} v_{\alpha} E_{\alpha} \in \mathbf{H}^{\hat{\otimes} n}(Y)$ we define

$$\mathbb{D}I_n(v) = n I_{n-1}(v(y, \cdot)), y \in U,$$

$I_0(c) = c, c \in \mathbf{R}$. For $u = \sum_n \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y)$ we define

$$\mathbb{D}u(y) = \sum_n \frac{\mathbb{D}I_n(u_n)}{n!} = \sum_n \frac{I_{n-1}(u_n(y, \cdot))}{(n-1)!}$$

We see that \mathbb{D} maps $\mathcal{S}(Y)$ into $\mathcal{S}(Y)$.

Remark 9. In the framework of a single r.v. ξ (see Remark 3), $\mathbb{D}^k(\xi^{\otimes n}) = \frac{n!}{(n-k)!} \xi^{\otimes(n-k)}, k \geq 1$.

3.3. Adapted stochastic processes. In this subsection we assume that $U = [0, T] \times V, \mathcal{U} = \mathcal{B}([0, T]) \times \mathcal{V}, d\mu = dt d\pi$. Let

$$(3.1) \quad u = \sum_n \frac{I_n(u_n)}{n!} \in \mathcal{S}'(Y),$$

with $u_n \in \mathbf{H}^{\hat{\otimes} n}(Y), n \geq 0$ ($u_n = u_n(t_1, v_1, \dots, t_n, v_n), (t_i, v_i) \in U, i = 1, \dots, n$).

For $t \in [0, T]$, let $Q_t^n = ([0, t] \times V)^n$.

Definition 6. Let $t_0 \in [0, T]$. A random variable u defined by (3.1) is called \mathcal{F}_{t_0} -measurable if for each n , $\text{supp}(u_n) \subseteq Q_{t_0}^n$, i.e. μ_{n-1} -a.e.

$$u_n(t_1, v_1, \dots, t_n, v_n) = 0 \text{ if } t_i > t_0 \text{ for some } i$$

Proposition 5. A random variable $u \in \mathcal{S}(Y)$ defined by (3.1) is \mathcal{F}_{t_0} -measurable iff

$$\mathbb{D}u(t, v) = 0 \text{ if } t > t_0.$$

Proof. For $u \in \mathcal{S}(Y)$ defined by (3.1),

$$(3.2) \quad \mathbb{D}u = \sum_{n \geq 1} \frac{1}{(n-1)!} I_{n-1}(u_n(t, v, \cdot), (t, v) \in U.$$

and

$$\begin{aligned} \mathbf{E} [\mathbb{D}u(t, v)^2] &= \sum_n \frac{1}{(n-1)!} \mathbf{E} [I_{n-1}(u_n(t, v, \cdot))^2] \\ &= \sum_n \int_{U^{n-1}} u_n(t, v, \cdot)^2 d\mu_{n-1} = 0 \end{aligned}$$

for $t > t_0$ iff $u_n(t, v, \cdot) = 0$ μ_{n-1} -a.e for $t > t_0$. \square

Now, we will introduce a notion of an adapted random process. Consider $u \in S(U; Y)$, i.e.,

$$(3.3) \quad u(t, v) = \sum_n \frac{I_n(u_n(t, v, \cdot))}{n!}$$

with $u_n(t, v, \cdot) \in \mathbf{H}^{\otimes n}(Y)$ for all $(t, v) \in U$:

$$(3.4) \quad u_n(t, v, \cdot) = \sum_{|\alpha|=n} u_\alpha(t, v) E_\alpha(\cdot), (t, v) \in U.$$

Definition 7. A random field $u(t, v)$ on U defined by (3.3) is called adapted if $\text{supp}(u_n(t, v, \cdot)) \subseteq Q_t^n$, $v \in V$, for every $t \in [0, T]$.

A straightforward consequence of Proposition 5 is the following claim.

Corollary 1. A random field $u \in S(U; Y)$ is adapted iff for each $t \in [0, T]$, the Malliavin derivative $\mathbb{D}u(t, v; s_1, v_1, \dots, s_n, v_n) = 0$, $v \in V$, if $s_i > t$ for some i .

Given a random field $u(t, v)$ on $U = [0, T] \times V$, consider its Skorokhod integral

$$\delta(u)_t = \delta(\chi_{[0, t]} u) = \int_0^t u(r, y) \diamond \mathfrak{N}(dr, dy), 0 \leq t \leq T.$$

Proposition 6. Consider a random field $u \in S(\mathbf{H}(Y); Y)$, i.e. (3.3) holds with

$$\int_{U^{n+1}} |u_n|_Y^2 d\mu_{n+1} < \infty \quad \forall n.$$

If it is adapted, then $\delta(u)_t$, $0 \leq t \leq T$, is adapted as well.

Proof. Since $u(t, v) = \sum_n \frac{I_n(u_n(t, v, \cdot))}{n!}$ is adapted with u_n satisfying (3.4), $\text{supp}(u_n(t, v, \cdot)) \subseteq Q_t^n, v \in V$, for all n and $t \in [0, T]$, i.e. $u_n(t, v) = u_n(t, v)\chi_{Q_t^n}$. By Proposition 3,

$$\delta(u)_t = \delta(\chi_{[0,t]}u) = \sum_{n=0}^{\infty} \frac{1}{n!} I_{n+1}(\widetilde{\chi_{[0,t]}u_n}),$$

where $\widetilde{\chi_{[0,t]}u_n}$ is the standard symmetrization of $\chi_{[0,t]}u_n = u_n\chi_{Q_t^{n+1}}$. Since its support is obviously a subset of Q_t^{n+1} , the statement follows. \square

3.4. Ito-Skorokhod isometry. Now we estimate the L_2 -norm of the Skorokhod integral.

Proposition 7. *Let $u = u(v) = \sum_{\alpha} u_{\alpha}(v) \mathfrak{N}_{\alpha} \in L_2(\mathbb{D}(U; Y), d\mu)$, i.e. $u_{\alpha} \in \mathbf{H}(Y)$:*

$$\int |u_{\alpha}(v)|_Y^2 d\mu < \infty, \alpha \in J,$$

with a finite number of $u_{\alpha} \neq 0$. Then

$$\mathbf{E} [|\delta(u)|_Y^2] = \mathbf{E} \left[\int_U |u(v)|_Y^2 d\mu \right] + \mathbf{E} \left[\int_{U^2} (\mathbb{D}u(v; v'), \mathbb{D}u(v'; v))_Y \mu(dv) \mu(dv') \right],$$

where

$$\mathbb{D}u(v; v') = \sum_{\alpha} \sum_{|p|=1} \frac{(\alpha + p)!}{\alpha!} u_{\alpha+p}(v) E_p(v') \mathfrak{N}_{\alpha}.$$

Proof. By definition,

$$\delta(u) = \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_U u_{\alpha-p}(x) E_p(x) d\mu \mathfrak{N}_{\alpha} = \sum_{\alpha} \sum_{|p|=1} \int_U u_{\alpha}(r) E_p(r) d\mu \mathfrak{N}_{\alpha+p}.$$

Hence

$$\begin{aligned} \mathbf{E} [|\delta(u)|_Y^2] &= \sum_{|\alpha| \geq 1} \left| \sum_{|p|=1} \int_U u_{\alpha-p}(x) E_p(x) d\mu \right|_Y^2 \alpha! \\ &= \sum_{|p|=|p'|=1} \sum_{|\alpha| \geq 1} \int_{U^2} (u_{\alpha-p}(x), u_{\alpha-p'}(x'))_Y E_p(x) E_{p'}(x') \mu(dx) \mu(dx') \alpha! \\ &= \sum_{p=p', |p|=1} \dots + \sum_{p \neq p', |p|=|p'|=1} \dots = A + B. \end{aligned}$$

Now

$$\begin{aligned}
A &= \sum_{|p|=1} \sum_{\alpha \geq p} \left| \int_U u_{\alpha-p}(x) E_p(x) d\mu \right|_Y^2 \alpha! = \sum_{\gamma} \sum_{|p|=1} \left| \int_U u_{\gamma}(x) E_p(x) d\mu \right|_Y^2 (\gamma + p)! \\
&= \sum_{\gamma} \sum_{|p|=1} \left| \int_U u_{\gamma}(x) E_p(x) d\mu \right|_Y^2 \gamma! + \sum_{\gamma \geq p} \sum_{|p|=1} \left| \int_U u_{\gamma}(x) E_p(x) d\mu \right|_Y^2 [(\gamma + p)! - \gamma!] \\
&= \sum_{\gamma} \int_U \gamma! |u_{\gamma}(x)|_Y^2 d\mu + \sum_{\gamma \geq p} \sum_{|p|=1} \left| \int_U u_{\gamma}(x) E_p(x) d\mu \right|_Y^2 [(\gamma + p)! - \gamma!].
\end{aligned}$$

Also,

$$\begin{aligned}
B &= \sum_{p \neq p', |p|=|p'|=1} \sum_{\alpha \geq p+p'} \int_{U^2} (u_{\alpha-p}(x), u_{\alpha-p'}(x'))_Y E_p(x) E_{p'}(x') \mu(dx) \mu(dx') \alpha! \\
&= \sum_{p \neq p', |p|=|p'|=1} \sum_{\beta \geq 0} \int_{U^2} (u_{\beta+p'}(x), u_{\beta+p}(x'))_Y \times \\
&\quad \times E_p(x) E_{p'}(x') \mu(dx) \mu(dx') (\beta + p + p')!.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbf{E} [\mathbb{D}u(x; x'), \mathbb{D}_x u(x'; x)]_Y \\
&= \sum_{\alpha} \sum_{|p|=|p'|=1} \frac{(\alpha + p)!}{\alpha!} (u_{\alpha+p}(x), a_{\alpha+p'}(x'))_Y \frac{(\alpha + p')!}{\alpha!} E_p(x') E_{p'}(x) \alpha! \\
&= \sum_{\alpha} \sum_{p=p', |p|=1} \dots + \sum_{\alpha} \sum_{p \neq p', |p|=|p'|=1} \dots = C + D.
\end{aligned}$$

Obviously,

$$C = \sum_{|p|=1} \sum_{\alpha \geq p} \frac{\alpha!}{(\alpha - p)!} (u_{\alpha}(x), u_{\alpha}(x'))_Y E_p(x) E_p(x') \alpha!.$$

Comparing

$$D = \sum_{\alpha} \sum_{p \neq p', |p|=|p'|=1} \frac{(\alpha + p)!}{\alpha!} (a_{\alpha+p}(x) E_p(x'), a_{\alpha+p'}(x') E_{p'}(x))_Y \frac{(\alpha + p')!}{\alpha!} \alpha!,$$

$\int_{U^2} C d\mu_2, \int_{U^2} D d\mu_2$ and A, B , the statement follows. \square

Corollary 2. *Let $u \in \mathcal{S}(\mathbf{H}(Y), Y)$, i.e. (3.3) holds with*

$$\int_{U^{n+1}} |u_n|_Y^2 d\mu_{n+1} < \infty \quad \forall n.$$

Then the statement of Proposition 7 holds for $\delta(u)$.

Proof. It is enough to prove the statement for $u(v) = I_n(u_n(v))$, where

$$u_n(v, \cdot) = \sum_{|\alpha|=n} u_{\alpha}(v) E_{\alpha}(\cdot)$$

with with a finite number nonzero $u_\alpha \in \mathbf{H}(Y)$:

$$\int_U |u_\alpha|_Y^2 d\mu < \infty.$$

In this case, $u = n! \sum_{|\alpha|=n} u_\alpha(v) \mathfrak{N}_\alpha \in L_2(\mathbb{D}(U; Y), d\mu)$ and Proposition 7 applies. We obtain the general case by linearity and passing to the limit. \square

Remark 10. *In the framework of a single r.v. ξ (see Remark 3), for $u = \sum_n u_n \xi^{\odot n}$ we have*

$$\mathbf{E} [\delta(u)^2] = \mathbf{E} [|u|^2] + \mathbf{E} [(\mathbb{D}u)^2].$$

For an adapted random field on $U = [0, T] \times V, d\mu = dt d\pi$, the standard isometry holds. It is an obvious consequence of Corollary 2

Corollary 3. *Let $\mathbf{H} = L^2([0, T] \times V, dt d\pi)$. Assume $u \in \mathcal{S}(\mathbf{H}(Y), Y)$ is an adapted random field on $U = [0, T] \times V$. Then*

$$\mathbf{E} [|\delta(u)|_Y^2] = \mathbf{E} \left[\int_U |u(t, v)|_Y^2 d\mu \right].$$

Duality between δ and \mathbb{D}

Proposition 8. *Let $u = u(x) = \sum_\alpha u_\alpha(x) \mathfrak{N}_\alpha \in L_2(\mathcal{D}(U; \mathbf{R}), d\mu)$, i.e.*

$$\int |u_\alpha(x)|^2 d\mu < \infty, \alpha \in J,$$

with a finite number of $u_\alpha \neq 0$. Let $v = \sum_\alpha v_\alpha \mathfrak{N}_\alpha \in \mathcal{D}$. Then

$$\mathbf{E}[\delta(u) v] = \mathbf{E} \left[\int_U u(x) \mathbb{D}v(x) d\mu \right].$$

Proof. Indeed,

$$\begin{aligned} & \mathbf{E}[\delta(u) v] \\ &= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_U u_{\alpha-p}(x) E_p(x) d\mu v_\alpha \alpha! \\ &= \sum_{|p|=1} \sum_{\alpha \geq p} \int_U u_{\alpha-p}(x) E_p(x) d\mu v_\alpha \alpha! \\ &= \sum_{|p|=1} \sum_{\gamma} \int_U u_\gamma(x) E_p(x) d\mu v_{\gamma+p} \frac{(\gamma+p)!}{\gamma!} \gamma! = \mathbf{E} \left[\int_U u(x) \mathbb{D}v(x) d\mu \right]. \end{aligned}$$

\square

4. STOCHASTIC DIFFERENTIAL EQUATIONS

4.0.1. *Wick exponent.* We start with the definition of Wick exponent.

Let $f = \sum_k f_k m_k \in L^2(U, d\mu)$. For $n \geq 1$ and $\mathfrak{N}(f) = \sum_k f_k \xi_k$ we have (denoting $f^\alpha = \prod_k f_k^{\alpha_k}$)

$$\mathfrak{N}(f)^{\diamond n} = \mathfrak{N}(f) \diamond \dots \diamond \mathfrak{N}(f) \text{ (n times)} = \sum_{|\alpha|=n} \frac{n!}{\alpha!} f^\alpha \mathfrak{N}_\alpha.$$

Note that $\mathfrak{N}(f)^{\diamond n} \in L^2(\Omega)$:

$$\begin{aligned} \mathbf{E} \left[\left(\frac{1}{n!} \mathfrak{N}(f)^{\diamond n} \right)^2 \right] &= \sum_{|\alpha|=n} \frac{z^{2\alpha}}{\alpha!} = \frac{1}{n!} \sum_{|\alpha|=n} \frac{n! z^{2\alpha}}{\alpha!} = \frac{1}{n!} \left(\sum_i z_i^2 \right)^n \\ &= \frac{1}{n!} |f|_{L^2(\mu)}^{2n} < \infty. \end{aligned}$$

Let \mathcal{Z} be the set of all number sequences $z = (z_k)$ with finite number of nonzero terms. The following statement holds.

Proposition 9. a) Let $f = \sum_k f_k m_k \in L^2(\mu)$. Then

$$\mathfrak{N}(f)^{\diamond n} = I_n(f^{\otimes n})$$

and

$$\exp^\diamond \{ \mathfrak{N}(f) \} := \sum_{n=0}^{\infty} \frac{\mathfrak{N}(f)^{\diamond n}}{n!} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} \mathfrak{N}_\alpha = \sum_{\alpha} \frac{f^\alpha}{\alpha!} \mathfrak{N}_\alpha \in L^2(\Omega)$$

with

$$\frac{f^\alpha}{\alpha!} = \frac{1}{n! \alpha!} \int f^{\otimes n} E_\alpha d\mu_n.$$

Moreover,

$$(4.1) \quad \mathbf{E} \left[(\exp^\diamond \{ \mathfrak{N}f \})^2 \right] = \exp \left\{ |f|_{L^2(\mu)}^2 \right\}.$$

b) Let $z = (z_k) \in \mathcal{Z}$. Then \mathbf{P} -a.s.

$$p(z) = \exp^\diamond \left\{ \mathfrak{N} \left(\sum_k z_k m_k \right) \right\} = \sum_{\alpha} \frac{z^\alpha}{\alpha!} \mathfrak{N}_\alpha, \quad z = (z_k) \in \mathcal{Z},$$

is analytic in z and

$$\frac{\partial^{|\alpha|} p(z)}{\partial z^\alpha} \Big|_{z=0} = \mathfrak{N}_\alpha.$$

Proof. a) In terms of multiple integrals we have

$$\begin{aligned} \mathfrak{N}(f)^{\diamond n} &= \sum_{|\alpha|=n} \frac{n! f^\alpha}{\alpha!} \mathfrak{N}_\alpha = \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} I_n(E_\alpha) = \\ &\quad \sum_{|p_i|=1, p_1+\dots+p_n=\alpha} \frac{n! f^\alpha}{\alpha!} I_n(\widetilde{E_{p_1} \dots E_{p_n}}) \\ &= I_n(f^{\otimes n}) = I_n \left(\sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} E_\alpha \right), \end{aligned}$$

with

$$f^{\otimes n} = \sum_{|\alpha|=n} \frac{f^\alpha}{\alpha!} E_\alpha, f^\alpha = \frac{1}{n!} \int f^{\otimes n} E_\alpha d\mu_n.$$

In addition,

$$\begin{aligned} \mathbf{E} \left[\left(\frac{\mathfrak{N}(f)^{\otimes n}}{n!} \right)^2 \right] &= \sum_{|\alpha|=n} \frac{f^{2\alpha}}{\alpha!} = \frac{1}{n!} \sum_{|\alpha|=n} \frac{n! f^{2\alpha}}{\alpha!} = \frac{1}{n!} \left(\sum_i f_i^2 \right)^n \\ &= \frac{1}{n!} |f|_{L^2(\mu)}^{2n} \end{aligned}$$

(note also using multiple integrals: $\mathbf{E} \left(\frac{1}{(n!)^2} I_n(f^{\otimes n})^2 \right) = \frac{1}{n!} |f^{\otimes n}|_{L^2(\mu_n)}^2 = \frac{1}{n!} |f|_{L^2(\mu)}^{2n}$). Moreover,

$$\mathbf{E} \left(\sum_n \left(\frac{\mathfrak{N}(f)^{\otimes n}}{n!} \right)^2 \right) = \sum_n \frac{1}{n!} |f|_{L^2(\mu)}^{2n} = \exp \left\{ |f|_{L^2(\mu)}^2 \right\}.$$

b) Let $z = (z_k) \in \mathcal{Z}$. Then

$$\mathbf{E} |p(z)| \leq \sum_\alpha \frac{|z^\alpha|}{\sqrt{\alpha!}} \mathbf{E} \left| \frac{\mathfrak{N}_\alpha}{\sqrt{\alpha!}} \right| \leq \sum_\alpha \frac{|z^\alpha|}{\sqrt{\alpha!}} \leq \prod_k \sum_n \frac{|z_k|^n}{\sqrt{n!}}$$

and the statement follows. \square

In a time dependent case the following statement holds.

Corollary 4. *Let $U = [0, T] \times V$, $d\mu = dt d\pi$. Let $G \in L^2([0, T] \times V, d\mu)$. Consider $M_t = \exp^\diamond \left\{ \mathfrak{N} \left(\chi_{[s, t]} G \right) \right\}$, $0 \leq s \leq t \leq T$.*

Then

$$M_t = \sum_\alpha \frac{H(s, t)^\alpha}{\alpha!} \mathfrak{N}_\alpha = \sum_{n=0}^{\infty} \frac{I_n(H_n(s, t))}{n!}, s \leq t \leq T,$$

with

$$\begin{aligned} H_k(s, t) &= \int_s^t \int G(r, v) m_k(r, v) dr d\pi, H(s, t)^\alpha = \prod_k H_k(s, t)^{\alpha_k}, \\ \frac{H(s, t)^\alpha}{\alpha!} &= \frac{1}{n! \alpha!} \int \left(\chi_{[s, t]} G \right)^{\otimes n} E_\alpha d\mu_n \text{ if } |\alpha| = n, \\ H_n(s, t) &= \left(\chi_{[s, t]} G \right)^{\otimes n}. \end{aligned}$$

Moreover, M is adapted (for each t , the $\text{supp}(H_n(t)) \subseteq Q_t^n = ([0, t] \times V)^n$).

4.0.2. *Linear SDE.* Let $U = [0, T] \times V$, $d\mu = dt d\pi$. Let $w = \sum_{\alpha} w_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'$, $f = \sum_{\alpha} f_{\alpha}(t, v) \mathfrak{N}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu)$. For $G \in L^2(\mu)$, consider the a non-homogeneous equation

$$(4.2) \quad \dot{u}(t) = \int [u(t) G(t, v) + f(t, v)] \diamond \mathfrak{N}(t, v) \pi(dv), u(0) = w,$$

that is, equivalently,

$$(4.3) \quad u(t) = w + \int_0^t [u(s) G(s, v) + f(s, v)] \diamond \mathfrak{N}(ds, dv), 0 \leq t \leq T.$$

We seek a solution to (4.3) in the form

$$(4.4) \quad u(t) = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha}, 0 \leq t \leq T.$$

Lemma 2. *Let $w = \sum_{\alpha} w_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'$, $f = \sum_{\alpha} f_{\alpha}(t, v) \mathfrak{N}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu)$. Then there is a unique solution to (4.3) in $C\mathcal{D}'([0, T]; \mathbf{R})$ (Recall $C\mathcal{D}'([0, T]; \mathbf{R})$ is the class of all generalized processes $u = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha}$ on $[0, T]$ such that u_{α} is continuous on $[0, T] \forall \alpha \in J$). The solution u given by (4.4) has the following coefficients: $u_0(t) = w_0$,*

$$(4.5) \quad u_{\alpha}(t) = w_{\alpha} + \sum_{|p|=1} \int_0^t \int [u_{\alpha-p}(r) G(r, v) + f_{\alpha-p}(r, v)] E_p(r, v) d\pi dr, 0 \leq t \leq T.$$

Proof. We seek the solution u to (4.3) in the form of (4.4) with continuous coefficients u_{α} . Plugging the series (4.4) into (4.3) we immediately get the system (4.5). Since the system is triangular, starting with $u_0(t) = w_0$ we find unique continuous $u_{\alpha}(t)$ for $|\alpha| \geq 1$. \square

Let

$$H_k(t) = \int_0^t \int G(s, v) m_k(s, v) ds d\pi, H(t)^{\alpha} = \prod_k H_k(t)^{\alpha_k}, \alpha \in J.$$

For $w = \sum_{\alpha} w_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'$, let

$$||w||^2 = \sum_{\alpha} |w_{\alpha}|^2 \alpha! + \sup_t \sum_{\alpha} \alpha! \left(\sum_{\beta \leq \alpha} w_{\alpha-\beta} \frac{H(t)^{\beta}}{\beta!} \right)^2$$

Lemma 3. *Let $f = 0$, $w = \sum_{\alpha} w_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'$.*

(i) *The solution to (4.3) is given by*

$$(4.6) \quad u(t) = w \diamond \exp^{\diamond} \left\{ \mathfrak{N} \left(\chi_{[0,t]} G \right) \right\} = \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha-\beta} \frac{H(r)^{\beta}}{\beta!} \mathfrak{N}_{\alpha}, 0 \leq r \leq T.$$

(ii) *If $w \in \mathcal{S}'(\mathbf{R})$, then $u \in C\mathcal{S}'([0, T]; \mathbf{R})$;*

(iii)

$$\sup_t \mathbf{E} [u(t)^2] \leq ||w||^2,$$

i.e. it is in $L^2(\Omega, \mathbf{P})$ if $\|w\| < \infty$ (see Example 4 below).
 (iv) If $w = w_0$ is a constant, then $u(t)$ is adapted and

$$\sup_t \mathbf{E} [u(t)^2] = w_0^2 \exp \left\{ \int |G|^2 d\mu \right\}.$$

Proof. (i) Let $M_t = \exp^\diamond \left\{ \mathfrak{N}(\chi_{[0,t]} G) \right\}$. By Corollary 4,

$$v(r) = w \diamond M_r = \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha-\beta} \frac{H(r)^\beta}{\beta!} \mathfrak{N}_\alpha, 0 \leq r \leq T.$$

We will show that v solves (4.3). Indeed,

$$\begin{aligned} & \delta \left(\chi_{[0,t]} v G \right) \\ &= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_0^t \int_V \sum_{\beta \leq \alpha-p} w_{\alpha-p-\beta} \frac{H(r)^\beta}{\beta!} G(r, v) E_p(r, v) d\mu \mathfrak{N}_\alpha \\ &= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_U \sum_{\beta \leq \alpha-p} w_{\alpha-p-\beta} \frac{\int \left(\chi_{[0,r]} G \right)^{\otimes |\beta|} E_\beta d\mu_{|\beta|}}{|\beta|! \beta!} \chi_{[0,t]}(r) G(r, v) E_p(r, v) d\mu \mathfrak{N}_\alpha \\ &= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_{U^{|\gamma|}} \sum_{\gamma \leq \alpha, |\gamma| \geq 1} w_{\alpha-\gamma} \left(\chi_{[0,r]} G \right)^{\otimes |\gamma|-1} \chi_{[0,t]}(r) G(r, v) \frac{E_\gamma}{(|\gamma|-1)! \gamma!} d\mu_{|\gamma|} \mathfrak{N}_\alpha \\ &= \sum_{|\alpha| \geq 1} \sum_{\gamma \leq \alpha, |\gamma| \geq 1} \int w_{\alpha-\gamma} \left(\chi_{[0,t]} G \right)^{\otimes |\gamma|} \frac{E_\gamma}{|\gamma|! \gamma!} d\mu_{|\gamma|} \mathfrak{N}_\alpha = v(t) - w, \end{aligned}$$

and (4.3) holds.

(ii) follows immediately by Lemma 7 and Proposition 9. The part (iii) is a direct consequence of (4.6). Finally, (iv) follows from (4.6), Proposition 9 and Corollary 4. \square

Example 4. Let $F \in L^2(U, d\mu)$. Taking $w = \exp^\diamond \{ \mathfrak{N}(F) \}$ in (4.6), we see that the solution to (4.3)

$$\begin{aligned} u(t) &= \exp^\diamond \{ \mathfrak{N}(F) \} \diamond \exp^\diamond \left\{ \mathfrak{N}(\chi_{[0,t]} G) \right\} \\ &= \exp^\diamond \left\{ \mathfrak{N}(F) + \mathfrak{N}(\chi_{[0,t]} G) \right\} \end{aligned}$$

is clearly non-adapted in general but $\sup_t \mathbf{E} [u(t)^2] < \infty$ (Proposition 9).

Let for $s \leq t$,

$$H_k(s, t) = \int \chi_{[s,t]} G m_k d\mu, H(s, t)^\alpha = \prod_k H_k(s, t)^{\alpha_k}, \alpha \in J.$$

For $f = \sum_{\alpha} f_{\alpha}(t, v) \mathfrak{N}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu)$, let $\|f\|_{0,T}^2 = \mathbf{E} \int_U |f|^2 d\mu$ and

$$\begin{aligned} \|f\|_T^2 &= \sup_t \sum_{\alpha} \alpha! \left(\sum_{|p|=1} \int_0^t \int_U \sum_{\beta+p \leq \alpha, |\beta| \leq n} f_{\alpha-p-\beta}(s, v) E_p(s, v) \frac{H(s, t)^{\beta}}{\beta!} d\mu \right)^2 \\ &\quad + \sum_{\alpha} \int_U |f_{\alpha}|^2 d\mu \alpha!. \end{aligned}$$

Proposition 10. *Let $f = \sum_{\alpha} f_{\alpha}(t, v) \mathfrak{N}_{\alpha} \in L_2(\mathcal{D}'(U), d\mu)$, $w = \sum_{\alpha} w_{\alpha} \mathfrak{N}_{\alpha} \in \mathcal{D}'$.*

(i) *The unique solution to (4.3) in $C\mathcal{D}'([0, T]; \mathbf{R})$ can be given as*

$$\begin{aligned} (4.7) \quad & \mathcal{U}(t) \\ &= w \diamond \exp^{\diamond} \left\{ \mathfrak{N}(\chi_{[0,t]} G) \right\} + \int_0^t \int \exp^{\diamond} \left\{ \mathfrak{N}(\chi_{[s,t]} G) \right\} \diamond f(s, v) \diamond \mathfrak{N}(ds, dv) \\ &= \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_0^t \int_U \sum_{\beta+p \leq \alpha, |\beta| \leq n} f_{\alpha-p-\beta}(s, v) E_p(s, v) \frac{H(s, t)^{\beta}}{\beta!} d\mu \mathfrak{N}_{\alpha} \\ &\quad + \sum_{\alpha} \sum_{\beta \leq \alpha} w_{\alpha-\beta} \frac{H(r)^{\beta}}{\beta!} \mathfrak{N}_{\alpha}. \end{aligned}$$

(ii) *The solution is the limit of Picards iterations $u^n(t)$: $u^0(t) = w + \int_0^t \int f(s, v) \diamond \mathfrak{N}(ds, dv)$,*

$$(4.8) \quad u^{n+1}(t) = w + \int_0^t \int [u^n(s) G(s, v) + f(s, v)] \diamond \mathfrak{N}(ds, dv), \quad 0 \leq t \leq T.$$

In fact,

$$(4.9) \quad u^n(t) = w \diamond \sum_{k=0}^n \frac{\mathfrak{N}(\chi_{[0,t]} G)^{\diamond k}}{k!} + \int_0^t \int \sum_{k=0}^n \frac{\mathfrak{N}(\chi_{[s,t]} G)^{\diamond k}}{k!} \diamond f(s, v) \diamond \mathfrak{N}(ds, dv).$$

If $f \in \mathcal{S}'(\mathbf{H}, \mathbf{R})$ and $w \in \mathcal{S}'(\mathbf{R})$, then $u^n, u \in C\mathcal{S}'([0, T]; \mathbf{R})$;

(iii)

$$\sup_t \mathbf{E} \left[u(t)^2 \right] \leq 2 \left(\|w\|^2 + \|f\|_T^2 \right).$$

(iv) *If $w = w_0$ is deterministic and f is adapted with $\|f\|_{0,T}^2 < \infty$, then $u(t)$ is adapted and square integrable:*

$$\sup_t \mathbf{E} \left[u(t)^2 \right] \leq C \left(w_0^2 + \mathbf{E} \int_U |f|^2 d\mu \right).$$

Proof. Because of Lemma 3 we assume $w = 0$.

(i) Let

$$\begin{aligned}
l(s, v) &= f(s, v) \diamond \exp^\diamond \left\{ \mathfrak{N}(\chi_{[s, r]}) G \right\} \\
&= \sum_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} f_{\alpha - \beta}(s, v) \frac{H(s, r)^\beta}{\beta!}, 0 \leq s \leq r \leq T \\
&= \sum_{\alpha} \sum_{\beta \leq \alpha, |\beta| \leq n} f_{\alpha - \beta}(s, v) \frac{1}{|\beta|! \beta!} \int \left(\chi_{[s, t]} G \right)^{\otimes |\beta|} E_\beta d\mu_{|\beta|} \mathfrak{N}_\alpha.
\end{aligned}$$

and set

$$\begin{aligned}
&v(r) \\
&= \int_0^r \int l(s, v) \diamond \mathfrak{N}(ds, dv) \\
&= \sum_{|\alpha| \geq 1} \sum_{|p|=1, p \leq \alpha} \int \sum_{\beta + p \leq \alpha, |\beta| \leq n} f_{\alpha - (p + \beta)}(s, v) E_p(s, v) d\mu \frac{H(s, t)^\beta}{\beta!} \mathfrak{N}_\alpha, 0 \leq r \leq T.
\end{aligned}$$

For $r \in [0, T]$, $v = (s_1, v_1, \dots, s_k, v_k) \in U^k$, $k \geq 1$, define

$$\begin{aligned}
\Phi(r, k, G, f) &= \Phi(r, k, G, f)(s_1, v_1, \dots, s_k, v_k) \\
&= \sum_{j=1}^k f(\hat{s}, v_j) \prod_{i \neq j, i=1}^k \chi_{[\hat{s}, r]}(s_i) G(s_i, v_i),
\end{aligned}$$

where $\hat{s} = \min \{s_i, 1 \leq i \leq k\}$.

By Corollary 4,

$$\begin{aligned}
&v(r) \\
&= \sum_{|\alpha| \geq 1, p \leq \alpha, |p|=1} \int_0^r \int \sum_{\beta + p \leq \alpha} f_{\alpha - (p + \beta)}(s, v) \frac{E_p(s, v)}{|\beta|! \beta!} \int \left(\chi_{[s, r]} G \right)^{\otimes |\beta|} E_\beta d\mu_{|\beta|} d\mu \mathfrak{N}_\alpha \\
&= \sum_{\alpha} \sum_{\beta' \leq \alpha, 1 \leq |\beta'|} \int_{U^{|\beta'|+1}} \left(\chi_{[s, r]} G \right)^{\otimes |\beta'| - 1} f_{\alpha - \beta'}(s, v) \frac{E_{\beta'}}{(|\beta'| - 1)! \beta'!} d\mu_{|\beta'|} \mathfrak{N}_\alpha \\
&= \sum_{\alpha} \sum_{\beta' \leq \alpha, 1 \leq |\beta'|} \int_{U^{|\beta'|}} \Phi(r, |\beta'|, G, f_{\alpha - \beta'}) \frac{E_{\beta'}}{|\beta'|! \beta'!} d\mu_{|\beta'|} \mathfrak{N}_\alpha, 0 \leq r \leq T.
\end{aligned}$$

We will show that v solves (4.3). Indeed,

$$\begin{aligned}
& \int_0^t v(r) G(r, v) \diamond \mathfrak{N}(dr, dv) \\
&= \sum_{|\alpha| \geq 2} \sum_{|p|=1} \sum_{\beta' + p \leq \alpha, 1 \leq |\beta'|} \int \chi_{[0, t]}(r) G(r, v) E_p(r, v) \times \\
& \quad \times \int_{U^{|\beta'|}} \Phi(r, |\beta'|, G, f_{\alpha - (p + \beta')}) \frac{E_{\beta'}}{|\beta'|! \beta'!} d\mu_{|\beta'|} d\mu \mathfrak{N}_\alpha \\
&= \sum_{|\alpha| \geq 2} \sum_{|p|=1} \sum_{\gamma = \beta' + p \leq \alpha, 2 \leq |\gamma| (\leq n+2)} \int_{U^{|\gamma|}} \chi_{[0, t]}(r) G(r, v) \Phi(r, |\gamma| - 1, G, f_{\alpha - \gamma}) \times \\
& \quad \times \frac{E_\gamma}{(|\gamma| - 1)! \gamma!} d\mu_{|\beta'|} d\mu \mathfrak{N}_\alpha \\
&= \sum_{|\alpha| \geq 2} \sum_{\gamma = \beta' + p \leq \alpha, 2 \leq |\gamma| (\leq n+2)} \int_{U^{|\gamma|}} \Phi(t, |\gamma|, G, f_{\alpha - \gamma}) \frac{E_\gamma}{|\gamma|! \gamma!} d\mu_{|\beta'|} d\mu \mathfrak{N}_\alpha
\end{aligned}$$

and we see that

$$\int_0^t v(r) G(r, v) \diamond \mathfrak{N}(dr, dv) = v(t) - \int_0^t \int f(r) \diamond G(r, v) \mathfrak{N}(dr, dv).$$

(ii) Consider $u^n(t)$ defined by (4.9) with $w = 0$. Then
(4.10)

$$u^n(t) = \sum_{|\alpha| \geq 1} \sum_{|p|=1, p \leq \alpha} \int \sum_{\beta + p \leq \alpha, |\beta| \leq n} f_{\alpha - (p + \beta)}(s, v) E_p(s, v) d\mu \frac{H(s, t)^\beta}{\beta!} \mathfrak{N}_\alpha, 0 \leq r \leq T,$$

and repeating the proof of part (i) we see that for $0 \leq t \leq T$

$$\int_0^t \int u^n(r) G(r, v) \diamond \mathfrak{N}(dr, dv) = u^{n+1}(t) - \int_0^t \int f(s, v) \diamond \mathfrak{N}(ds, dv).$$

If $f \in \mathcal{S}'(\mathbf{H}, \mathbf{R})$, then $u^0 \in C\mathcal{S}'([0, T]; \mathbf{R})$. If $u^n \in C\mathcal{S}'([0, T]; \mathbf{R})$, then $u^n G \in \mathcal{S}'(\mathbf{H}, \mathbf{R})$. By Proposition 3, $u^{n+1} \in C\mathcal{S}'([0, T]; \mathbf{R})$ and the statement follows by comparing (4.10) and (4.7).

The part (iii) is a direct consequence of (4.7).

(iv) Since $\mathbf{E} \int |f|^2 d\mu < \infty$, it follows that $f \in \mathcal{S}'(\mathbf{H}, \mathbf{R})$, and according to part (ii) and Proposition 6, all the iterations are adapted. Therefore Ito isometry holds. Obviously,

$$\sup_t \mathbf{E} [u^0(t)^2] \leq \mathbf{E} \int_U |f|^2 d\mu < \infty.$$

Assume $\sup_t \mathbf{E} [u^n(t)^2] < \infty$. Using (4.8) and Ito isometry,

$$\mathbf{E} [u^{n+1}(t)^2] \leq C \left[\int_0^t \int \mathbf{E} (u^n(s)^2) G(s, v)^2 d\mu ds + \mathbf{E} \int_0^t \int |f|^2 ds d\pi \right]$$

and by Gronwall's lemma there is a constant C independent of n such that

$$\sup_{n,t} \mathbf{E} \left(u^n(t)^2 \right) \leq C \mathbf{E} \int_0^T \int |f|^2 ds d\pi.$$

Similarly, using Gronwall's lemma, we show that

$$\sum_n \sup_t \mathbf{E} \left([u^{n+1}(t) - u^n(t)]^2 \right) < \infty.$$

The statement follows. \square

4.0.3. Linear parabolic SPDEs. In this section we extend the results on the linear SDE to a simple parabolic SPDE.

Again, let $U = [0, T] \times V, d\mu = dt d\pi$. We denote $\mathbf{R}_T^d = \mathbf{R}^d \times [0, T]$ and suppose that the following measurable functions are given

$$a : \mathbf{R}^d \rightarrow \mathbf{R}^{d^2}, \quad b : \mathbf{R}^d \rightarrow \mathbf{R}^d.$$

The following is assumed.

A1. Functions a, b , are infinitely differentiable and bounded with all derivatives, and the matrix $a = (a^{ij}(x))$ is symmetric and non-degenerate: for all x

$$a^{ij}(x) \xi_i \xi_j \geq \delta |\xi|^2, \xi \in \mathbf{R}^d,$$

for some $\delta > 0$.

Let $H_2^s = H_2^s(\mathbf{R}^d)$, $s = 1, 2$, be the Sobolev class of square-integrable functions v on \mathbf{R}^d having generalized space derivatives up to the s -order with the finite norm

$$|v|_{s,2} = |v|_2 + |D_x^s v|_2,$$

where $|v|_2 = (\int_{\mathbf{R}^d} |v|^2 dx)^{1/2}$.

Let $G \in L^2([0, T] \times V, d\mu)$ with $d\mu = dt d\pi$. Let $L^{2,1} = L^{2,1}(\mathbf{R}^d \times [0, T] \times V, dx dt d\pi)$ be the space of all measurable functions g on $\mathbf{R}^d \times [0, T] \times V$ such that

$$\|g\|_{1,2}^2 = \int_0^T \int_{\mathbf{R}^d} \int_V [|g(s, x, v)|^2 + |\nabla_x g(s, x, v)|^2] ds dx d\pi < \infty.$$

Let $w = \sum_\alpha w_\alpha(x) \mathfrak{N}_\alpha \in \mathcal{D}'(H_2^3(\mathbf{R}^d))$ and $f = \sum_\alpha f_\alpha(x, s, v) \mathfrak{N}_\alpha \in \mathcal{D}'(L^{2,1})$. The main objective of this section is to study the equation for $u(t) = u(t, x)$,

$$\begin{aligned} (4.11) \quad \partial_t u(x, t) &= \mathcal{L}u(x, t) \\ &+ \int_U (u(x, t) G(t, v) + f(x, t, v)) \diamond \dot{\mathfrak{N}}(t, v) \pi(dv) \\ u(0, x) &= \varphi(x), \end{aligned}$$

where $\mathcal{L}u = a^{ij}(x) u_{x_i x_j} + b^i(x) u_{x_i}$. Equivalently, we understand (4.11) as

$$\begin{aligned} (4.12) \quad u(t) &= w + \int_0^t \mathcal{L}u(s) ds \\ &+ \int_0^t \int_U [u(s) G(s, v) + f(s, v)] \diamond \mathfrak{N}(ds, dv), \end{aligned}$$

$0 \leq t \leq T$.

We will seek a solution to (4.12) in the form

$$(4.13) \quad u(t) = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha} \in C\mathcal{D}'([0, T]; H_2^2).$$

We start our analysis of equation (4.12) by introducing the definition of a solution in the "weak sense".

Definition 8. We say that a generalized \mathcal{D} -process $u(t) = \sum_{\alpha} u_{\alpha}(t) \xi_{\alpha} \in C\mathcal{D}'([0, T], H_2^2)$ is \mathcal{D} - H_2^2 solution of equation (4.12) in $[0, T]$, if the equality (4.12) holds in $\mathcal{D}(L^2(\mathbf{R}^d))$ for every $0 \leq t \leq T$.

Lemma 4. Assume **A1** holds, $w(x) = \sum_{\alpha} w_{\alpha}(x) \mathfrak{N}_{\alpha} \in \mathcal{D}'(H_2^3)$,

$g = \sum_{\alpha} g_{\alpha}(x, s, v) \mathfrak{N}_{\alpha} \in \mathcal{D}'(L^{2,1})$. Then there is a unique solution to (4.12) in $C\mathcal{D}'([0, T], H_2^2)$ (Recall $C\mathcal{D}'([0, T], H_2^2)$ is the class of all generalized processes $u = \sum_{\alpha} u_{\alpha}(t) \mathfrak{N}_{\alpha}$ on $[0, T]$ such that u_{α} is H_2^2 -valued continuous on $[0, T] \forall \alpha \in J$). The solution u given by (4.13) has the following coefficients: $u_0(t) = w_0$,

$$(4.14) \quad \begin{cases} \partial_t u_{\alpha}(t) = \mathcal{L}u_{\alpha} + \sum_k \int_V m_k(u_{\alpha(k)} G + f_{\alpha(k)}) d\pi \\ u_{\alpha}(0) = w_{\alpha}. \end{cases}$$

Proof. We seek the solution u to (4.12) in the form of (4.13) with continuous coefficients u_{α} . Plugging the series (4.13) into (4.12) we immediately get the system (4.14). Indeed, by definition, for $t \in [0, T]$,

$$\begin{aligned} \sum_{\alpha} u_{\alpha}(x, t) \mathfrak{N}_{\alpha} &= \sum_{\alpha} w_{\alpha}(x) \mathfrak{N}_{\alpha} \\ &+ \sum_{\alpha} \int_0^t \mathcal{L}u_{\alpha}(x, s) ds \mathfrak{N}_{\alpha} + \sum_{\alpha} \sum_k \int_0^t \int_V m_k[u_{\alpha(k)} G + g_{\alpha(k)}] d\pi ds \mathfrak{N}_{\alpha}. \end{aligned}$$

Since the system is triangular, starting with $u_0(t) = w_0$ we find unique continuous $u_{\alpha}(t)$ for $|\alpha| \geq 1$ (see [2]). \square

Denote by $T_t f$ the solution of the problem

$$\begin{cases} \partial_t u = \mathcal{L}u, & 0 \leq t \leq T, \\ u(0, x) = h(x), & x \in \mathbf{R}^d. \end{cases}$$

Remark 11. If **A1** holds, then it is well known that

$$(4.15) \quad |T_t h|_{L^2(\mathbf{R}^d)}^2 \leq e^{Ct} |h|_{L^2(\mathbf{R}^d)}^2,$$

(see [2]).

Note that for each α , the solution u_{α} of (4.14) satisfies for $t \in [0, T]$,

$$(4.16) \quad \begin{aligned} u_{\alpha}(t) &= T_t w_{\alpha} + \sum_k \int_0^t \int_V [m_k(s, v) (T_{t-s} u_{\alpha(k)}(s) G(s, v) \\ &+ T_{t-s} g_{\alpha(k)}(s, v))] ds dv. \end{aligned}$$

Lemma 5. Assume A1 holds,

$$w(x) = \sum_{\alpha} w_{\alpha}(x) \mathfrak{N}_{\alpha} \in \mathcal{D}'(H_2^3), g = \sum_{\alpha} g_{\alpha}(x, s, v) \mathfrak{N}_{\alpha} \in \mathcal{D}'(L^{2,1}).$$

Then u is the unique solution to (4.12) in $C\mathcal{D}'([0, T], H_2^2)$ iff it is the unique solution to

$$(4.17) \quad u(t) = \int_0^t \int_U [T_{t-s}u(s)G(s, v) + T_{t-s}g(s, v)] \diamond \mathfrak{N}(ds, dv) + T_t w,$$

$$0 \leq t \leq T.$$

Proof. Since (4.16) holds, the statement is an immediate consequence of Lemma 4. \square

The following statement holds.

Proposition 11. Let A1 hold and $w = \sum_{\alpha} w_{\alpha}(x) \mathfrak{N}_{\alpha} \in \mathcal{D}'(H_2^2(\mathbf{R}^d))$ and $g = \sum_{\alpha} f_{\alpha}(x, s, v) \mathfrak{N}_{\alpha} \in \mathcal{D}'(L^{2,1})$.

(i) The unique solution to (4.12) is given by

$$u(t) = T_t w(x) \diamond \exp^{\diamond} \left\{ \mathfrak{N} \left(\chi_{[0,t]} G \right) \right\} + \int_0^t \int_U \exp^{\diamond} \left\{ \mathfrak{N} \left(\chi_{[s,t]} G \right) \right\} \diamond T_{t-s} g(s, x, v) \diamond \mathfrak{N}(ds, dv)$$

In the form of the series,

$$u(t) = \sum_{|\alpha| \geq 1} \sum_{|p|=1} \int_0^t \int_U \sum_{\beta+p \leq \alpha} T_{t-s} f_{\alpha-p-\beta}(s, v) E_p(s, v) \frac{H(s, t)^{\beta}}{\beta!} d\mu \mathfrak{N}_{\alpha} + \sum_{\alpha} \sum_{\beta \leq \alpha} T_t w_{\alpha-\beta} \frac{H(t)^{\beta}}{\beta!} \mathfrak{N}_{\alpha}.$$

(ii) The solution is the limit of Picards iterations $u^n(t)$: $u^0(t) = T_t w + \int_0^t \int_U T_{t-s} f(s, v) \diamond \mathfrak{N}(ds, dv)$,

$$u^{n+1}(t) = T_t w + \int_0^t \int_U [T_{t-s} u^n(s) G(s, v) + T_{t-s} f(s, v)] \diamond \mathfrak{N}(ds, dv),$$

$0 \leq t \leq T$. In fact, for $0 \leq t \leq T$,

$$u^n(t) = T_t w \diamond \sum_{k=0}^n \frac{\mathfrak{N} \left(\chi_{[0,t]} G \right)^{\diamond k}}{k!} + \int_0^t \int_U \sum_{k=0}^n \frac{\mathfrak{N} \left(\chi_{[s,t]} G \right)^{\diamond k}}{k!} \diamond T_{t-s} f(s, v) \diamond \mathfrak{N}(ds, dv).$$

If $f \in \mathcal{S}'(\mathbf{H}, \mathbf{R})$ and $w \in \mathcal{S}'(\mathbf{R})$, then $u^n, u \in CS'([0, T]; \mathbf{R})$;

(iii) If w is deterministic and g is adapted, then the solution u is $L^2(\mathbf{R}^d)$ -valued and

$$\sup_t \mathbf{E} \left[|u(t)|_{L^2(\mathbf{R}^d)}^2 \right] \leq C \mathbf{E}[|w|_{L^2(\mathbf{R}^d)}] + \int_0^T \int_{\mathbf{R}^d} \int_U |g(x, s, v)|^2 ds dx d\pi.$$

Proof. We repeat the main arguments of Proposition 10 (as in the case of linear SDE). The changes in the proof (i) are obvious. The proof of (ii)-(iii) is identical to the proof of (ii), (iv) in Proposition 10 with the use of (4.15) for the estimate of the iterations $L^2(\Omega, \mathbf{P})$ -norm. \square

4.1. Stationary SPDEs. Let us consider a stationary (time independent) equation

$$(4.18) \quad \mathbf{A}u + \delta_{\mathfrak{N}}(\mathbf{M}u) = g$$

where, as previously, \mathfrak{N} -noise is a formal series $\mathfrak{N} = \sum_k m_k \xi_k$ and $\{m_k\}$ is a CONS in a Hilbert space H and ξ_k are independent random variables with zero mean and variance 1.

We will consider equation (4.18) in a triple of Hilbert spaces (V, H, V') ,

- A. $V \subset H \subset V'$ and the imbeddings $V \subset H$ and $H \subset V'$ are dense and continuous;
- B. The space V' is dual to V relative to the inner product in H ;
- C. There exists a constant $C > 0$ such that $|(u, v)_H| \leq C \|u\|_V \|v\|_{V'}$ for all u and v .

A triple of Hilbert spaces is often call *normal* if the assumptions A,B,C hold. A typical example of a normal triple is the Sobolev spaces

$$\left(H^{l+\gamma}(\mathbb{R}^d), H^l(\mathbb{R}^d), H^{l-\gamma}(\mathbb{R}^d) \right) \text{ for } \gamma > 0.$$

Everywhere in this section it is assumed that $\mathbf{A} : V \rightarrow V'$ and $\mathbf{M} : V \rightarrow V' \otimes l_2$ are bounded linear operators.

As we already know, equation (4.18) can be rewritten in the form

$$(4.19) \quad \mathbf{A}u + \sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n = f,$$

where $u = \sum_{\alpha} u_{\alpha} \mathfrak{N}_{\alpha}$. Since

$$(4.20) \quad \mathbf{M}_n u = \sum_{\alpha \in \mathcal{J}} \mathbf{M}_n u_{\alpha} \mathfrak{N}_{\alpha},$$

we get

$$\begin{aligned} & \sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n \\ &= \sum_{\alpha \in \mathcal{J}} \sum_{n \geq 1} \mathbf{M}_n u_{\alpha} \xi_{\alpha} \diamond \xi_n + \sum_{\alpha \in \mathcal{J}} \sum_{n \geq 1} \mathbf{M}_n u_{\alpha} \mathfrak{N}_{\alpha + \varepsilon_n} \\ &= \sum_{n \geq 1} \sum_{\beta \in \mathcal{J}: |\beta| \geq 1} \mathbf{M}_n u_{\beta - \varepsilon_n} \mathfrak{N}_{\beta}. \end{aligned}$$

Therefore, for $\gamma \in J$ such that $|\gamma| > 0$, we have

$$\left(\sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n \right)_\gamma = \sum_{n \geq 1} \mathbf{M}_n u_{\gamma - \varepsilon_n}$$

It is readily checked that the set $(u_\alpha, \alpha \in J)$ solves the following system of deterministic equations related to (4.19) is given by

$$(4.21) \quad \begin{aligned} \mathbf{A}u_\alpha &= Ef \quad \text{if } |\alpha| = 0 \\ \mathbf{A}u_\alpha + \sum_{n \geq 1} \mathbf{M}_n u_{\alpha - \varepsilon_n} &= f_\gamma \quad \text{if } |\alpha| > 0 \end{aligned}$$

Note that the propagator (4.21) is lower triangular. Therefore, if A has an appropriate inverse A^{-1} , then the propagator can be solved sequentially. Then, a solution to equation (4.19) could be defined by the following formula

$$(4.22) \quad u = \sum_{\alpha \in J} u_\alpha \mathfrak{N}_\alpha$$

where the sequence $\{u_\alpha, \alpha \in J\}$.

Of course, an appropriate question to ask is: does equation (4.19) has finite variance? The answer to this question is negative. The following simple example clarifies this issue.

Example 5. . Consider the following simple version of equation

$$(4.23) \quad u = 1 + u \diamond \xi.$$

Obviously, in this setting, $J = (0, 1, 2, \dots)$ and consists of one-dimensional indices $\alpha = 0, 1, 2, \dots$ and $\mathbf{E}[\mathfrak{N}_i^2] = i!$,

It is easy to see that $\{u_n = \mathbb{E}(u \mathfrak{N}_n), n \geq 0\}$ solves the following system of equations:

$$u_0 = 1, \quad u_n = I_{n=0} + \sqrt{n} u_{n-1}, \quad n \geq 1$$

Obviously, $u_n = \sqrt{n!}$ and $v = 1 + \sqrt{n!} \mathfrak{N}_n$, Therefore,

$$\mathbb{E}u^2 = \sum_{n \geq 1} u_n^2 = \infty.$$

One could also define a solution to equation (4.19) as a generalized \mathcal{D} -random variable with values in V , such that (4.19) holds in $\mathcal{D}(V')$.

4.1.1. *Weighted Norms.* Another popular definition of solutions based on

rescaling/weighting of the coefficients u_α was discussed thoroughly in the literature on polynomial chaos expansion for Gaussian and Levy processes (see, for example, [11], [3], [8], [10]). This technique is also suitable for the current setting and we will describe it briefly.

Given a separable Hilbert space X and sequence of positive numbers $R = \{r_\alpha, \alpha \in J\}$, we define the space $RL_2(X)$ as the collection of formal series $f = \sum_\alpha f_\alpha \mathfrak{N}_\alpha$, $f_\alpha \in X$ such that

$$(4.24) \quad \|f\|_{RL_2(X)}^2 = \sum_\alpha \|f_\alpha\|_X^2 r_\alpha^2 < \infty$$

If (4.24) holds, then $\sum_\alpha r_\alpha f_\alpha \xi_\alpha \in L_2(X)$.

Similarly, the space $R^{-1}L_2(X)$ corresponds to the sequence $R^{-1} = \{r_\alpha^{-1}, \alpha \in J\}$.

Important and popular examples of the space $RL_2(X)$ correspond to the following weights:

(a) $r_\alpha^2 = \prod_{k=1}^\infty q_k^{\alpha_k}$, where $\{q_k, k \geq 1\}$ is a non-increasing sequence of positive numbers;

(b) Kondratiev's spaces $(S)_{\rho, \alpha}$:

$$r_\alpha^2 = (\alpha!)^\rho (2\mathbb{N})^{l\alpha} \quad \rho \leq 0, \quad l \leq 0.$$

In particular, in the setting of Example 5, $\mathbb{E}u^2 = \|u\|_{(S)_{0,0}}^2 = \infty$, but $E\|u\|_{(S)_{\rho,0}}^2 < \infty$ for sufficiently small ρ .

4.1.2. *Wick-Nonlinear SPDEs.* Let us consider equation

$$(4.25) \quad \mathbf{A}u - u^{\diamond 3} + \sum_{n \geq 1} \mathbf{M}_n u \diamond \xi_n = f,$$

where $u^{\diamond 3} = u \diamond u \diamond u$. As in the previous section, we will look for a chaos solution of the form

$$u = \sum_{\kappa \in J} u_\kappa \mathfrak{N}_\kappa.$$

Obviously,

$$u^{\diamond 3} = \sum_{\kappa, \beta, \gamma \in J} u_\kappa u_\beta u_\gamma \mathfrak{N}_{\kappa+\beta+\gamma}.$$

Therefore,

$$(u^{\diamond 3})_\alpha = \sum_{\kappa, \beta, \gamma: \kappa+\beta+\gamma=\alpha} u_\kappa u_\beta u_\gamma$$

and the propagator of equation (4.25) is given by

$$(4.26) \quad \mathbf{A}u_\alpha - \sum_{\kappa, \beta, \gamma: \kappa+\beta+\gamma=\alpha} u_\kappa u_\beta u_\gamma + \sum_{n \geq 1} \mathbf{M}_n u_{\alpha-\varepsilon_n} = f_\alpha$$

for all $\alpha \in J$.

Similarly to (4.21), system (4.26) is also lower triangular and could be solved sequentially, assuming that operator A has an appropriate inverse.

It is readily checked that if the Wick cubic $u^{\diamond 3}$ is replaced by any Wick type polynomial then the related propagator system remains to be lower triangular.

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5. APPENDIX

Let (ξ_k) be a sequence of r.v.. We assume that the following assumption holds.

G. For each vector r.v. $(\xi_{i_1}, \dots, \xi_{i_n})$ the moment generating function

$$M_{i_1 \dots i_n}(t) = M_{i_1 \dots i_n}(t_1, \dots, t_n) = \mathbf{E} \exp \{t_1 \xi_{i_1} + \dots t_n \xi_{i_n}\}$$

exists for all $t = (t_1, \dots, t_n)$ in some neighborhood of $0 \in \mathbf{R}^n$.

Denote \mathcal{J} the set of all multiindices $\alpha = (\alpha_1, \alpha_2, \dots)$ such that $|\alpha| < \infty$ and $\alpha_k \in \{0, 1, 2, \dots\}$. Let $\mathcal{G} = \sigma(\xi_k, k \geq 1)$.

Lemma 6. *Let **G** holds. Then every $f \in L^2(\mathcal{G}, \mathbf{P})$ can be approximated in $L^2(\mathcal{G}, \mathbf{P})$ by a sequence of polynomials in $\xi^\alpha = \prod_k \xi_k^{\alpha_k}, \alpha \in \mathcal{J}$.*

Proof. The assumption **G** implies that all the moments of $\xi_k, k \geq 1$, exist. Assume $f \in L^2(\mathcal{G}, \mathbf{P})$ and

$$(5.1) \quad \mathbf{E}[f \xi^\alpha] = 0 \quad \forall \alpha \in \mathcal{J}.$$

It is enough to show that $f = 0$ a.s. in such a case.

Let $\xi^n = (\xi_1, \dots, \xi_n), \theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$. Then

$$h(r, \theta) = \mathbf{E}[\exp \{r \theta \cdot \xi^n\} f]$$

exists for $r \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Since $h(r)$ is a bilateral Laplace transform, by Theorem 5a, p. 57 in [17], it must exist and be analytic for all complex values of r in the strip $-\varepsilon < \operatorname{Re} r < \varepsilon$. In addition, because of (5.1),

$$h(\iota u, \theta) = \mathbf{E}[\exp \{\iota u \theta \cdot \xi^n\} f] = 0$$

for all $u \in \mathbf{R}$ (here $\iota^2 = -1$). In particular,

$$\tilde{h}(\theta) = \mathbf{E}[\exp \{\iota \theta \cdot \xi^n\} f] = 0$$

for all $\theta \in \mathbf{R}^n$. Therefore,

$$(5.2) \quad \mathbf{E}[g(\xi^n) f] = 0$$

for any continuous periodic function on \mathbf{R}^n . Then approximating with long period functions we see that (5.2) holds for a continuous function on \mathbf{R}^n with compact support, and then for any continuous bounded function g as well. Since n is arbitrary, it follows that $f = 0$ a.s. \square

Lemma 7. *For $v = \sum_{|\alpha|=n} v_\alpha E_\alpha \in \mathbf{H}^{\hat{\otimes} n}(Y)$ and $u = \sum_{|\alpha|=n} u_\alpha E_\alpha \in \mathbf{H}^{\hat{\otimes} n}(Y)$, we have*

$$I_n(v) \diamond I_m(u) = I_{n+m}(\widetilde{v \otimes_Y u}),$$

where $v \otimes_Y u = (v(x), u(y))_Y, x \in U^n, y \in U^m$.

Proof. Indeed, $I_n(v) = n! \sum_{|\alpha|=n} v_\alpha \mathfrak{N}_\alpha$, $I_m(u) = m! \sum_{|\alpha|=m} u_\alpha \mathfrak{N}_\alpha$ and

$$\begin{aligned}
I_n(v) \diamond I_m(u) &= n!m! \sum_{\alpha} \sum_{\beta \leq \alpha} (v_\beta, u_{\alpha-\beta}) \mathfrak{N}_\alpha \\
&= n!m! \sum_{|\alpha|=n+m} \sum_{\beta \leq \alpha} \int (v \otimes_Y u) \frac{E_\beta}{\beta!n!} \frac{E_{\alpha-\beta}}{(\alpha-\beta)!m!} (v_\beta, u_{\alpha-\beta}) \mathfrak{N}_\alpha \\
&= \sum_{\alpha} \sum_{\beta \leq \alpha} \int (v \otimes_Y u) \frac{E_\beta}{\beta!} \frac{E_{\alpha-\beta}}{(\alpha-\beta)!} \mathfrak{N}_\alpha = \sum_{\alpha} \int (v \otimes_Y u) \frac{E_\alpha}{\alpha!} \mathfrak{N}_\alpha \\
&= (n+m)! \sum_{|\alpha|=n+m} l_\alpha \mathfrak{N}_\alpha = I_{n+m} \left(\widetilde{v \otimes_Y u} \right),
\end{aligned}$$

because $\widetilde{v \otimes_Y u} = \sum_{|\alpha|=n+m} l_\alpha E_\alpha$ with

$$l_\alpha = \frac{1}{\alpha!(n+m)!} \int (v \otimes_Y u) E_\alpha d\mu_{n+m}.$$

□

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